NUMERICAL ANALYSIS PROJECT
PART I: ORDINARY DIFFERENTIAL EQUATIONS

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1 Introduction

The project consists in carrying out the tasks given and record all the results in a written report, ideally using Latex. Attention should be paid to:

1. Correctness of the results.

2. Presentation of the results and layout of the report.

3. Critical discussion of the results using the studied theory on numerical methods.

The contents of the report will then be discussed (orally) with the examiners, at which the student may be asked to explain details of the programs and to re-run some of the problems. The project is *individual* and counts towards 40% of the final mark. The project should be handed in electronic form (zip archive file) to the lecturers, calling the file: `surname_name_tel2007_8_project.zip`. The archive must contain the project report (in pdf) and the related programs written in MATLAB. Other accepted programming languages are SCILAB, Octave, FORTRAN, C, C++ and BASIC.

2 RL circuit
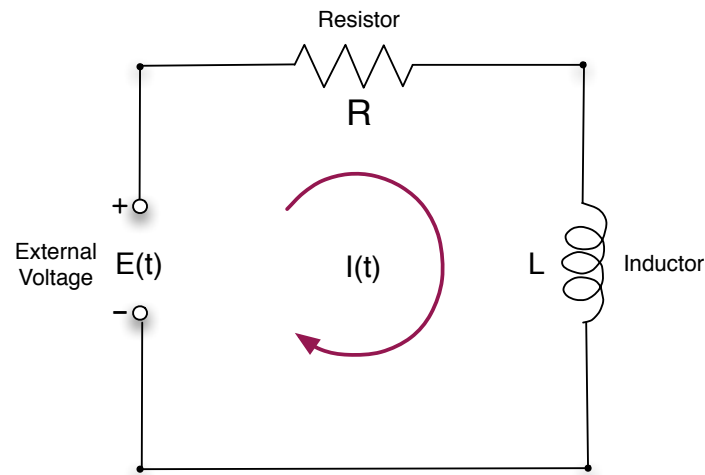
Consider the ordinary differential equation

$$I'(t) = -\frac{R}{L}I(t) + \frac{E}{L} \ , \tag{1}$$

with initial condition

$$I(0) = I_0 : \ \text{known value} \ , \tag{2}$$

where $I(t)$ is the instantaneous current (ampers), R is resistance (ohms) L is inductance and $E(t)$ is electromotive force. Assume $E = E_0$: constant then the exact solution of (1)-(2) is

$$I(t) = Ce^{-\alpha t} + \frac{E_0}{R} \ ,$$

with

$$\alpha = \frac{R}{L} \ , \qquad C = I_0 - \frac{E_0}{R} \ .$$

Problem 1. Assuming the values $R = 1$, $L = 1$ and $E_0 = 1$ and solve (1)-(2) and applying the following methods in the time interval $[0, 10]$

- The Euler method

- The second-order Runge-Kutta method

- The fourth-order Runge-Kutta method

- The implicit Euler method

- The implicit trapezoidal method

perform the following tasks

- Estimate numerically the order of convergence

- For each method find the optimal time step.

- Run each method with its optimal time step and plot the numerical solution and the exact solution in a single graph.

- Plot all the numerical solutions and the exact solution in a single graph.

3 RLC-circuit
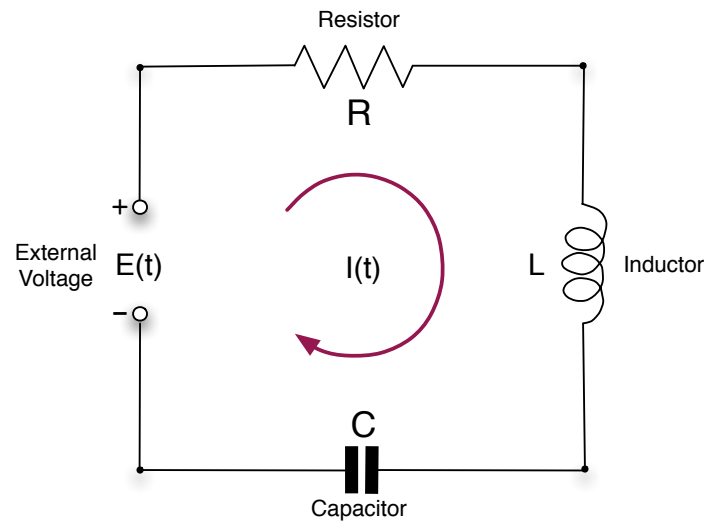
In an RLC-circuit we have a resistor R (ohms), an inductor L (henrys), a capacitor C (farads) and an source of electromotive source $E(t)$ and the following voltage drops:

$$\left.\begin{array}{l} E_L = LI' \text{ , across the inductor} \\ E_R = RI \text{ , across the resistor} \\ E_C = \frac{1}{C} \int I(t)dt \text{ , across the capacitor} \end{array}\right\}$$

Using Kirchhoffs's law one obtains the second order linear ordinary differential equation

$$I''(t) + \frac{R}{L}I'(t) + \frac{1}{LC}I(t) = \frac{1}{L}E(t) . \tag{3}$$

Consider the following situation:

$$\left.\begin{array}{l} E(t) = E_0 \sin(\omega t) \text{ electromotive force} \\ I(0) = 0 \text{ initial condition at time } 0 \end{array}\right\}$$

and the following parameters

$$\left.\begin{array}{l} R = 100 \text{ ohms} \\ L = 0.1 \text{ henry} \\ C = 10^{-3} \text{ farad} \\ \omega = 377 \\ E_0 = 155 \end{array}\right\}$$

The exact solution is given by

$$I(t) = c_1 e^{-10.10205140\,t} + c_2 e^{-989.8979486\,t} + c_3 \cos(377\,t) + c_4 \sin(377\,t)$$

$$c_1 = 0.005213795953, \qquad c_2 = -0.001552158513,$$

$$c_3 = -0.003661637439, \qquad c_4 = -0.001283311653$$

Tasks:

- For the explicit methods: second-order Runge-Kutta, fourth-order Runge-Kutta, solve the problem in the time interval $[0, 1]$ and compare results with the exact solution

- For the methods above find the optimal time step, solve the problem with the found optimal time stem and compare results with the exact solution again.

- Solve the problem using the implicit Euler method and the Trapezoidal method for various choices of the time step and compare results with the exact solution.

- Solve the problem using some "package", compare with the exact solution and compare the CPU times.

- Choose an error at the output time $t = 1$, $e = 10^{-6}$, say. Run each method with the appropriate time step until the preset error is attained and display the corresponding computing times employed. Which method is the most efficient ?

4   Nonlinear problem

The van der Pol problem originates from electronics and describes the behaviour of nonlinear vacuum tube circuits. The circuit scheme, designed by Balthazar van der Pol in the 1920's, is given in Figure
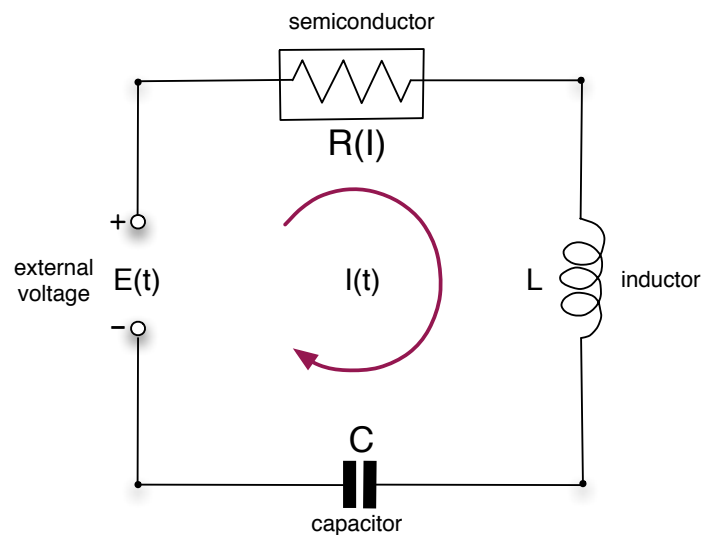
This is an RLC loop, but with the passive resistor of Ohm's Law replaced by an active element which would pump energy into the circuit whenever the amplitude of the current falls below a certain level. In the 1920's this active element was an array of vacuum tubes, now it is a semiconductor device. B. van der Pol assumed the nonlinear resistor to be represented by the nonlinear function $R(I) = aI^2 - b$ and substituting $R(I)$ for R in (3) the current in the circuit turns out to be modeled by

$$I''(t) + \frac{aI(t)^2 - b}{L}I'(t) + \frac{1}{LC}I(t) = \frac{1}{L}E(t) . \tag{4}$$

scaling the time $t = \tau/\sqrt{LC}$ and substituting in equation (4) becomes

$$I''(\tau) + (aI(\tau)^2 - b)\sqrt{\frac{C}{L}}I'(\tau) + I(\tau) = CE(\tau) .$$

setting $I(\tau) = z(\tau)\sqrt{b/a}$ and $\mu = b\sqrt{C/L}$

$$z''(\tau) + \mu(z(\tau)^2 - 1)z'(\tau) + z(\tau) = \sqrt{\frac{a}{b}}CE(\tau) .$$

setting $E(\tau) \equiv 0$ and call $\tau = t$ the van der Pol equation takes the standard form:

$$z''(\tau) + \mu(z(\tau)^2 - 1)z'(\tau) + z(\tau) = 0 . \tag{5}$$

This is a standard example of very stiff equation when μ becomes large. Consider now the equation (5) with initial data $z(0) = 2$ and $z'(0) = 0$.
Tasks:

- For $\mu = 1, 10, 1000, 10000$ use the MATLAB ODE solvers to compute the numerical solution in the interval $[0, 2\mu]$. What happen when μ becomes large ?

- Solve the problem using the implicit Euler method and the Trapezoidal method for various choices of the time step and compare results with the solution obtained with the MATLAB ODE solvers.