Numerical analysis course.
Telecommunications Engineering.
Project 2 for the period June 2008 to February 2009.

This project requires the implementation of several numerical methods studied in the course using a programming language of your preference (eg. FORTRAN, C, etc). For some of the problems you may simply use programmes from the Library NUMERICA (E. F. Toro, 1999), or you may want to translate them to the programming language of your choice.

A written report on the project is required, which should be delivered in the form of a pdf file to professor Toro directly (email: toro@ing.unitn.it).

Please name your pdf file as follows: TL-initial.surname-06.08-02.09.pdf

Example: TL-R.Conci-06.08-02.09.pdf

Note that his project remains valid for the period June 2008 to February 2009.

1 Problem 1: linear advection equation

Consider two test problems for the linear advection equation

$$\partial_t q(x,t) + \lambda \partial_x q(x,t) = 0 \;, \lambda = 1 \;. \tag{1}$$

The spatial domain is defined alongside the initial conditions for each test problem.

Test 1: The initial condition is smooth and is given by

$$q(x,0) = e^{-8x^2} \;, \quad -1 < x < 1 \;. \tag{2}$$

Test 2: The initial condition is discontinuous and is given as

$$u(x,0) = \begin{cases} 0 & \text{if} \quad x \le 0.3 \;, \\ 1 & \text{if} \quad 0.3 \le x \le 0.7 \;, \\ 0 & \text{if} \quad x \ge 0.7 \;. \end{cases} \tag{3}$$

Task 1: linear advection equation. Compute numerical solutions to above two test problems using the following numerical methods

- The first-order upwind method of Godunov;
- The Lax-Friedrichs method;
- The FORCE method;
- The Lax-Wendroff method;
- The MUSCL-Hancock with the superbee limiter;
- The MUSCL-Hancock with ENO reconstruction;
- The ADER2 with the superbee limiter;
- The ADER2 with ENO reconstruction.

For each test problem use the mesh of $M = 100$ cells, periodic boundary conditions, CFL coefficient $C_{\text{cfl}} = 0.9$ and the output time $t = 10$.

- For each method and test problem plot the numerical solution and the exact solution at the required output time;
- Comment on the numerical results using the theory studied in the course.

2 Problem 2: Electrical transmission line

Consider a transmission line with positive constant electrical parameters:

- C is the capacitance to ground per unit length,
- G is the conductance to ground per unit length,
- R is the resistance per unit length,
- L is the inductance per unit length.

The problem is to determine the current $I(x,t)$ and the potential $E(x,t)$ as functions of space and time. These quantities are solutions of the system

$$\left.\begin{aligned} \partial_t I(x,t) + \tfrac{1}{L}\partial_x E(x,t) &= -\tfrac{R}{L} I \\ \partial_t E(x,t) + \tfrac{1}{C}\partial_x I(x,t) &= -\tfrac{G}{C} E \end{aligned}\right\} \tag{4}$$

Assume $C = G = R = L = 1$, for simplicity.

Task 1: analysis

- Find the eigenvalues of the system;
- Find the right eigenvectors;
- Find the left eigenvectors;
- Write the system in terms of characteristic variables;
- Suppose that the initial distribution of the current $I(x,0)$ and the potential $E(x,0)$ are known functions of distance. Using the canonical form of the equations (or equations in characteristic variables) find the exact solution of the general initial-value problem for the homogeneous version of (4) (no source terms) and then transform back to obtain the solution in terms of the original variables;
- Solve the Riemann problem for the homogeneous version (no source terms) of equations (4) in which the initial conditions are

$$\left.\begin{aligned} I(x,0) &= \begin{cases} I_L & \text{if} \quad x < 0\,, \\ I_R & \text{if} \quad x > 0\,, \end{cases} \\ E(x,0) &= \begin{cases} E_L & \text{if} \quad x < 0\,, \\ E_R & \text{if} \quad x > 0\,. \end{cases} \end{aligned}\right\} \tag{5}$$

Task 2: Specific Riemann problem. Consider the homogeneous Riemann problem for (4) in a spatial domain $[0, 10]$, with the following initial conditions

$$\left.\begin{aligned} I(x,0) &= \begin{cases} I_L = 1 & \text{if} \quad x < 5\,, \\ I_R = 1/2 & \text{if} \quad x > 5\,, \end{cases} \\ E(x,0) &= \begin{cases} E_L = 1/4 & \text{if} \quad x < 5\,, \\ E_R = 3/4 & \text{if} \quad x > 5\,. \end{cases} \end{aligned}\right\} \tag{6}$$

Solve this problem exactly and plot the solution profiles for $I(x,\hat{t})$ and $E(x,\hat{t})$ for $\hat{t} = 1, 2$.

Task 3: numerics including source terms. Solve numerically the same problem up to time $\hat{t} = 2$ using the following numerical methods for the equations (4) without the source terms:

- The first-order Godunov method with the exact Riemann solver;
- The FORCE flux;
- The ADER2 with the exact Riemann solver, with ENO reconstruction in terms of the conserved variables;
- The ADER2 with the exact Riemann solver, with ENO reconstruction in terms of the characteristic variables.

For all methods compare the numerical solution (plot with symbols) with the exact solution (plotted with a full line). Comment on results.

Task 4: numerics including source terms. Implement the following schemes for the above equations including source terms

- The MUSCL-Hancock method with the exact Riemann solver, with ENO reconstruction in terms of the characteristic variables;
- The ADER2 method with the exact Riemann solver, with ENO reconstruction in terms of the characteristic variables.

Comment on results.

3 Written report

Write a report on the above tasks, with a well-structured layout (introduction, the main body of the work using graphics, conclusions, references, etc). The following criteria will be used to mark the project:

- Presentation, structure and lay out of contents (15%)
- Correctness of the results (20%)
- Interpretation and discussion of results (15%)
- Oral defense of the project (50%)