Numerical Analysis, Tel. Engineering. 9th February 2009.

Question 1. Consider the linear advection equation

$$\partial_t q(x,t) + \lambda \partial_x q(x,t) = 0 \tag{1}$$

with $\lambda = 1$, along with the initial condition

$$q(s,0) = h(s) = \begin{cases} 0 & \text{if} \quad s < -1\,, \\ 1 - s^2 & \text{if} \quad -1 \le s \le 1\,, \\ 0 & \text{if} \quad s > 1\,. \end{cases} \tag{2}$$

(a) Plot the solution $q(x,t) = h(x - \lambda t)$ of the initial value problem problem (1), (2) at times $t = 0, 1, 2$.

(b) Replot $q(x,t)$ in the three-dimensional x-t-q space.

Question 2. Consider the three-point linear scheme

$$q_i^{n+1} = b_{-1} q_{i-1}^n + b_1 q_i^n + b_1 q_{i+1}^n \tag{3}$$

for the linear advection equation

$$\partial_t q(x,t) + \lambda \partial_x q(x,t) = 0 \tag{4}$$

with constant λ. There is a theoretical result that says that (3) is linearly stable if and only if the following two conditions are satisfied: (i) $b_0(b_{-1} + b_1) \ge 0$ and (ii) $b_0(b_{-1} + b_1) + 4b_{-1}b_1 \ge 0$.

(a) Apply the above result to analyse the linear stability of the Godunov centred method, for which $b_{-1} = \frac{1}{2}c(1 + 2c)$; $b_0 = 1 - 2c^2$; $b_1 = -\frac{1}{2}c(1 - 2c)$.

(b) Analise the monotonicity of the scheme for Courant numbers c in the range of linear stability. Comment on your findings.

(c) Use the above theoretical result to show that any monotone three-point linear scheme (3) is linearly stable.

Question 3. Consider the model, abstract, hyperbolic system

$$\partial_t \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} + \begin{bmatrix} 0 & a \\ a & 0 \end{bmatrix} \partial_x \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{5}$$

or in matrix form

$$\partial_t \mathbf{Q} + \mathbf{A} \partial_x \mathbf{Q} = \mathbf{0}\,, \tag{6}$$

with the obvious notation for the vector $\mathbf{Q}$ of unknowns and the coefficient matrix $\mathbf{A}$. Find the eigenstructure of the system, that is:

(a) Find the eigenvalues of the system.

(b) Find the left eigenvectors, with scaling factors α_1, α_2.

(c) Find the right eigenvectors with scaling factors β_1, β_2.

(d) Find the correct scaling for α_1, α_2, β_1, β_2 such that the left $\mathbf{L}^{(i)}$ and right $\mathbf{R}^{(j)}$ eigenvectors are orthonormal, that is $\mathbf{L}^{(i)} \cdot \mathbf{L}^{(j)} = 1$ if $i = j$ and 0, otherwise.