Numerical Analysis. Laurea Magistrale, Telecommunications Engineering.
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Question 1. Consider the boundary value problem (BVP)

$$\left.\begin{array}{ll} \text{ODE:} & q''(t) = s(t, q(t), q'(t)) \equiv c(t)q' + d(t)q + e(t) \ , \quad t \in [0, b] \ , \\ & \\ \text{BCs:} & q(0) = \alpha \ , q(b) = \beta \ , \end{array}\right\} \tag{1}$$

where $c(t)$, $d(t)$, $e(t)$ are known continuous functions of t. Discretize the interval $[0, b]$ with $N + 1$ equidistant points ($N - 1$ internal points) and by approximating derivatives by finite differences formulate the boundary-value problem for finding the solution $q(t)$ at the points $t^1, t^2, t^3, \ldots, t^{N-1}$ in terms of a linear system of algebraic equations of the form $\mathbf{AX} = \mathbf{B}$, specifying the matrix $\mathbf{A}$ and the vectors $\mathbf{X}$ and $\mathbf{B}$. Justify every step of your answer.

Question 2. Consider the homogeneous linear advection equation with constant coefficient

$$\partial_t q + \lambda \partial_x q = 0 \ , \quad \lambda : \ \text{Constant} \ , \quad x \in (-\infty, \infty) \ , \tag{2}$$

with initial condition

$$q(x, 0) = h(x) = \left\{ \begin{array}{ccc} 0 & \text{if} & x < -1 \ , \\ \frac{1}{2}(1 + \cos(\pi x)) & \text{if} & -1 \leq x \leq 1 \ , \\ 0 & \text{if} & x > 1 \ . \end{array} \right. \tag{3}$$

- (a) For $\lambda = 1$ and $t = 2$ determine the solution $q(x, 2)$ by drawing a table of values of $q(x, 2) = h(\hat{X})$, with $\hat{X} = x - \lambda t$, for appropriate values of x. Plot the profile.
- (b) Repeat the above procedure for $\lambda = -1$ by plotting the profile of $q(x, 2)$.

Question 3. Consider the linear system with constant coefficients

$$\partial_t \begin{bmatrix} q_1 \\ \\ q_2 \end{bmatrix} + \begin{bmatrix} 0 & -a \\ \\ -a & 0 \end{bmatrix} \partial_x \begin{bmatrix} q_1 \\ \\ q_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \\ 0 \end{bmatrix} \ , \tag{4}$$

or

$$\partial_t \mathbf{Q} + \mathbf{A} \partial_x \mathbf{Q} = \mathbf{0} \ , \tag{5}$$

with the obvious notation. Here $a > 0$ is a real number.

- (a) Find the eigenvalues of the system, the right eigenvectors, with scaling factors α_1, α_2 and the left eigenvectors with scaling factors γ_1, γ_2.
- (b) By taking $\alpha_1 = 1$, $\alpha_2 = 1$, find γ_1 and γ_2 such that the left and right eigenvectors are orthonormal, that is $\mathbf{L}_i \ . \ \mathbf{R}_j = 1$, if $i = j$, and 0, otherwise.
- (c) With the scaling in (b) above find the matrix $\mathbf{R}$ of right eigenvectors, the inverse matrix $\mathbf{R}^{-1}$ and verify that $\mathbf{L} = \mathbf{R}^{-1}$, where $\mathbf{L}$ is the matrix of left eigenvectors.
- (d) Find the characteristic variables $\mathbf{C} = \mathbf{R}^{-1}\mathbf{Q}$.
- (e) Find the complete solution of the Riemann problem for the model linear system with initial conditions

$$\left.\begin{array}{l} q_1(x, 0) = \left\{ \begin{array}{lll} q_{1,L} & \text{if} & x < 0 \ , \\ q_{1,R} & \text{if} & x > 0 \ , \end{array} \right. \\ \\ q_2(x, 0) = \left\{ \begin{array}{lll} q_{2,L} & \text{if} & x < 0 \ , \\ q_{2,R} & \text{if} & x > 0 \ . \end{array} \right. \end{array}\right\} \tag{6}$$

Question 4. Consider the model advection-reaction equation

$$\partial_t q + \lambda \partial_x q = \beta q \ , \tag{7}$$

where λ and β are two constants, with $\beta < 0$. Consider the numerical scheme to solve the above PDE

$$q_i^{n+1} = q_i^n - \frac{1}{2} c (q_{i+1}^n - q_{i-1}^n) + r q_i^n \ , \tag{8}$$

with $c = \lambda \Delta t / \Delta x$ and $r = \beta \Delta t$.

- (a) Write the scheme in the form $q_i^{n+1} = \sum_{k=-1}^{k=1} b_k q_{i+k}^n$ and analyze the monotonicity of the scheme
- (b) Analyze the linear stability of the scheme using the von Neumann method. Comment on your findings.

Question 5.

Consider the elliptic Boundary Value Problem

$$\left.\begin{array}{l} \partial_x^{(2)} q(x,y) + \partial_y^{(2)} q(x,y) = 0 \ , \ (x,y) \in [0,1] \times [0,1] \ , \\ q(x,0) = q_B(x) = 1 - x \ , \ x \in [0,1] \ , \\ q(x,b) = q_T(x) = 1 - x \ , \ x \in [0,1] \ , \\ q(0,y) = q_L(y) = 1 \ , \ y \in [0,1] \ , \\ q(a,y) = q_R(y) = 0 \ , \ y \in [0,1] \ . \end{array}\right\} \tag{9}$$

Consider the five-point formula

$$q_{i,j} = \frac{q_{i-1,j} + q_{i+1,j} + q_{i,j-1} + q_{i,j+1}}{4} \ , \tag{10}$$

with $\Delta x = \Delta y = 1/3$.

- (a) Illustrate through a drawing the geometric situation, clearly identifying the vector of unknowns $\mathbf{X}$.
- (b) Assuming guess values $x_l^{(0)} = 1/2$, for $l = 1, 2, 3, 4$, apply the Jacobi method for 3 iterations, measuring the norm of the error $\mathbf{E}^{(k+1)} = ||\mathbf{X}^{(k+1)} - \mathbf{X}^{(k)}||$ at each iteration k. As a norm of a vector $\mathbf{V} = [v_1, v_2, \ldots, v_m]$ use $||\mathbf{V}|| = max_l(|v_l|)$. Put all results on a table (all calculations to be made by hand).