Numerical Analysis. Laurea Magistrale, Telecommunications Engineering.
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Question 1. Consider the model IVP

$$\left.\begin{array}{ll} \text{ODE:} & q'(t) = \beta q(t)\ , t \in [0,1]\ , \beta \leq 0\ , \\ \text{IC:} & q(0) = 1 \end{array}\right\} \qquad (1)$$

and the Taylor method of order p to solve it:

$$\left.\begin{array}{l} q^{n+1} = q^n + h\phi(t^n, q^n, h)\ , \\ \phi(t^n, q^n, h) = q^{(1)}(t^n) + \frac{1}{2}h^1 q^{(2)}(t^n) + \ldots + \frac{1}{p!}h^{p-1}q^{(n)}(t^n)\ . \end{array}\right\} \qquad (2)$$

Find the local truncation error for the case $p = 2$.

Question 2. Consider a model for an infinite transmission line given by the PDEs

$$\left.\begin{array}{l} \partial_t I(x,t) + \frac{1}{L}\partial_x E(x,t) = -\frac{R}{L} I\ , \\ \\ \partial_t E(x,t) + \frac{1}{C}\partial_x I(x,t) = -\frac{G}{C} E\ . \end{array}\right\} \qquad (3)$$

where the unknowns are I and E. Assume the source terms are identically zero.

- (a) Find the eigenvalues of the system and consider them in increasing order.
- (b) Find the right eigenvectors R_i with scaling factors α_1, α_2.
- (c) Find the left eigenvectors L_i with scaling factors γ_1, γ_2.
- (d) Assuming $\alpha_1 = 1$, $\alpha_2 = 1$, find the scaling for γ_1, γ_2 such that the left and right eigenvectors are orthonormal, that is $L_i.R_j = 1$ if $i = j$ e $L_i.R_j = 0$ if $i \neq j$.
- (e) Find the characteristic variables.

Question 3. Recall that the finite volume formulation for non-linear systems in one space dimension with source terms

$$\partial_t q + \partial_x f(q) = s(q)\ , \qquad (4)$$

has the form

$$q_i^{n+1} = q_i^n - \frac{\Delta t}{\Delta x}[f_{i+\frac{1}{2}} - f_{i-\frac{1}{2}}] + \Delta t s_i\ . \qquad (5)$$

Recall also that each term is to be interpreted as follows

$$q_i^n = \frac{1}{\Delta x}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} q(x, t^n)dx\ , \qquad (6)$$

$$f_{i+\frac{1}{2}} = \frac{1}{\Delta t}\int_0^{\Delta t} f(q(x_{i+\frac{1}{2}}, t))dt\ , \qquad (7)$$

and

$$s_i = \frac{1}{\Delta t}\frac{1}{\Delta x}\int_0^{\Delta t}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} s(q_i(x,t))dxdt\ . \qquad (8)$$

Consider in particular the model advection-reaction equation

$$\partial_t q + \lambda\partial_x q = \beta q\ , f(q) = \lambda q; s(q) = \beta q\ , \qquad (9)$$

where $\lambda > 0$ and $\beta \leq 0$ are two constants Consider a numerical scheme constructed as follows: (i) for the numerical flux use Godunov upwind method, that is, evaluate the flux integral at the interface $x_{i+\frac{1}{2}}$ by assuming a constant integrand in (7) equal to λq_i^n (for $\lambda > 0$); (ii) for the numerical source evaluate the source double integral (8) by assuming a "frozen" value $\frac{1}{2}(q_{i-1}^n + q_i^n)$ for $q_i(x,t)$.

- (a) verify that the resulting numerical scheme for (9) is

$$q_i^{n+1} = q_i^n - c(q_i^n - q_{i-1}^n) + \frac{1}{2} r(q_{i-1}^n + q_i^n) \; . \tag{10}$$

- (b) write the scheme in the form $q_i^{n+1} = \sum_{k=-l}^{k=1} b_k q_{i+k}^n$ and analyze the monotonicity of the scheme for $\lambda > 0$.

- (c) analyze the linear stability of the scheme using the von Neumann method. Discuss your results.

Question 4. Consider the initial-boundary problem

$$\left.\begin{array}{ll} \text{PDE:} & \partial_t q(x,t) = \alpha \partial_x^{(2)} q(x,t) \; , \;\; x \in (0,b) \; , \;\; t > 0 \; , \\ \\ \text{IC:} & q(x,0) = q^{(0)}(x) \; , \;\; x \in (0,b) \; , \\ \\ \text{BCs:} & q(0,t) = q_0(t) \; , \;\; q(b,t) = q_b(t) \; , \; t \geq 0 \; . \end{array}\right\} \tag{11}$$

Take $b = 1$, $\alpha = 1$ and initial condition

$$q(x,0) = \begin{cases} q_L = 1 & \text{if} \quad x < 1/2 \; , \\ \\ q_R = 0 & \text{if} \quad x > 1/2 \; , \end{cases} \tag{12}$$

As boundary conditions take $q_0(t) = 1$ and $q_b(t) = 0$. Discretize the domain by a finite difference mesh with M=4 interior points.

- (a) Apply "by hand" the explicit FTCS scheme with $D = 0.4$ for 5 time steps and construct a table of solution values for each time step.

- (b) Construct the implicit FTCS scheme for the same problem and write the resulting linear system in matrix form (do not solve it).

Question 5. Consider the elliptic Boundary Value Problem

$$\left.\begin{array}{l} \partial_x^{(2)} q(x,y) + \partial_y^{(2)} q(x,y) = 0 \; , \;\; (x,y) \in [0,1] \times [0,1] \; , \\ \\ q(x,0) = q_B(x) = 1 - x \; , \;\; x \in [0,1] \; , \\ \\ q(x,b) = q_T(x) = 1 - x \; , \;\; x \in [0,1] \; , \\ \\ q(0,y) = q_L(y) = 1 \; , \;\; y \in [0,1] \; , \\ \\ q(a,y) = q_R(y) = 0 \; , \;\; y \in [0,1] \end{array}\right\} \tag{13}$$

and the finite difference five-point formula

$$q_{i,j} = \frac{q_{i-1,j} + q_{i+1,j} + q_{i,j-1} + q_{i,j+1}}{4} \; , \tag{14}$$

with $\Delta x = \Delta y = 1/3$.

- (a) Illustrate through a drawing the geometric situation (domain, discretization and notation), clearly identifying the vector of unknowns $\mathbf{X}$.

- (b) Assuming guess values $x_l^{(0)} = 1/4$, for $l = 1, 2, 3, 4$, apply the Jacobi method for 3 iterations, measuring the norm of the error $\mathbf{E}^{(k+1)} = ||\mathbf{X}^{(k+1)} - \mathbf{X}^{(k)}||$ at each iteration k. As a norm of a vector $\mathbf{V} = [v_1, v_2, \ldots, v_m]$ use $||\mathbf{V}|| = max_l(|v_l|)$. Put all results on a table (all calculations to be made by hand).