

Tables and summary

“Numerical Methods for Dynamic System and Control”

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Fourier Serie

$$S_f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k \cos \frac{k\pi x}{\ell} + b_k \sin \frac{k\pi x}{\ell} \right),$$

$$a_0 = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) dx,$$

$$a_k = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \cos \frac{k\pi x}{\ell} dx,$$

$$b_k = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \sin \frac{k\pi x}{\ell} dx.$$

Z and Laplace transform

Z TRANSFORM TABLE: $\sum_{k=0}^{\infty} f_k z^{-k}$	
$k_{\ell} = k(k-1) \cdots (k-\ell+1) = \frac{k!}{(k-\ell)!}$	
δ_k	1
$\mathbf{1}_k$	$\frac{z}{z-1}$
a^k	$\frac{z}{z-a}$

ka^k	$\frac{za}{(z-a)^2}$
$k(k-1)a^k$	$\frac{2za^2}{(z-a)^3}$
$k_\ell a^k$	$\ell! \frac{za^\ell}{(z-a)^{\ell+1}}$
$a^k \binom{k}{\ell}$	$\frac{za^\ell}{(z-a)^{\ell+1}}$
$a^k f_k$	$\tilde{f}\left(\frac{z}{a}\right)$
kf_k	$-z \frac{d\tilde{f}(z)}{dz}$
$k^2 f_k$	$\left(z \frac{d}{dz}\right) \left(z \frac{d}{dz}\right) \tilde{f}(z)$
$k^\ell f_k$	$\left(-z \frac{d}{dz}\right)^\ell \tilde{f}(z)$
$f_{k+\ell}$	$z^\ell \left(\tilde{f}(z) - \sum_{j=0}^{\ell-1} f_j z^{-j}\right)$
$f_{k-\ell}$	$z^{-\ell} \tilde{f}(z)$
$k_\ell f_{k-\ell}$	$(-1)^\ell z \frac{d^\ell}{dz^\ell} \left(\frac{1}{z} \tilde{f}(z)\right)$
$(f \star g)_k$	$\tilde{f}(z) \tilde{g}(z)$
$a^k \sin \omega k$	$\frac{za \sin \omega}{z^2 - 2za \cos \omega + a^2}$
...	$\frac{za \sin \omega}{(z - a \cos \omega)^2 + (a \sin \omega)^2}$
$a^k \cos \omega k$	$\frac{z^2 - za \cos \omega}{z^2 - 2za \cos \omega + a^2}$
...	$\frac{z^2 - za \cos \omega}{(z - a \cos \omega)^2 + (a \sin \omega)^2}$

LAPLACE TRANSFORM TABLE $\int_0^{\infty} f(t)e^{-st} dt$	
$a f(t) + b g(t)$	$a \widehat{f}(s) + b \widehat{g}(s)$
$f(at)$	$\frac{1}{a} \widehat{f}\left(\frac{s}{a}\right) \quad [a > 0]$
$e^{at} f(t)$	$\widehat{f}(s - a)$
$f(t - a)$	$e^{-as} \widehat{f}(s)$
$1, t, t^k$	$\frac{1}{s}, \frac{1}{s^2}, \frac{k!}{s^{k+1}}$
a^{bt}	$\frac{1}{s - b \log a}$
$\int_0^t f(z) dz$	$\frac{1}{s} \widehat{f}(s)$
$f'(t)$	$s \widehat{f}(s) - f(0^+)$
$f''(t)$	$s^2 \widehat{f}(s) - f'(0^+) - s f(0^+)$
$\frac{d^n}{dt^n} f(t)$	$s^n \widehat{f}(s) - \sum_{j=0}^{n-1} s^{n-j-1} f^{(j)}(0^+)$
$t^n f(t)$	$(-1)^n \frac{d^n}{ds^n} \widehat{f}(s)$
$(f \star g)(t)$	$\widehat{f}(s) \widehat{g}(s)$
$e^{at} \cos \omega t, \quad e^{at} \sin \omega t$	$\frac{s - a}{(s - a)^2 + \omega^2}, \quad \frac{\omega}{(s - a)^2 + \omega^2}$
$e^{at} \cosh \omega t, \quad e^{at} \sinh \omega t$	$\frac{s - a}{(s - a)^2 - \omega^2}, \quad \frac{\omega}{(s - a)^2 - \omega^2}$
$e^{at} t^n$	$\frac{n!}{(s - a)^{n+1}}$
$e^{\alpha t} - e^{\beta t}$	$\frac{\alpha - \beta}{(s - \alpha)(s - \beta)}$

Constrained minima and Lagrange multiplier

Consider the constrained minimization problem

$$\begin{aligned} \text{minimize:} & \quad f(\mathbf{x}) \\ \text{subject to:} & \quad h_i(\mathbf{x}) = 0 \quad i = 1, 2, \dots, m \end{aligned}$$

Solution algorithm

- Compute the Lagrangian function: $\mathcal{L}(\mathbf{x}, \boldsymbol{\lambda}) = f(\mathbf{x}) - \sum_{k=1}^m \lambda_k h_k(\mathbf{x})$
- Solve the nonlinear system $\nabla_{\mathbf{x}} \mathcal{L}(\mathbf{x}, \boldsymbol{\lambda}) = \mathbf{0}^T$ with $\mathbf{h}(\mathbf{x}) = \mathbf{0}$.
- For each solution points $(\mathbf{x}^*, \boldsymbol{\lambda}^*)$ compute $\nabla \mathbf{h}(\mathbf{x}^*)$ and check it is full rank, or the rows are linearly independent.
- Compute the matrix \mathbf{K} the kernel of $\nabla \mathbf{h}(\mathbf{x}^*)$, i.e. $\nabla \mathbf{h}(\mathbf{x}^*) \mathbf{K} = \mathbf{0}$.
- Compute the reduce Hessian

$$\mathbf{H} = \mathbf{K}^T \nabla_{\mathbf{x}}^2 \mathcal{L}(\mathbf{x}^*, \boldsymbol{\lambda}^*) \mathbf{K},$$

- Necessary condition: \mathbf{H} is semi-positive definite.
- Sufficient condition: \mathbf{H} is positive definite.

The following theorem prove the sufficient condition.

Theorem 1 (of Lagrange multiplier) Let $f \in C^2(\mathbb{R}^n, \mathbb{R})$ a map and \mathbf{x}^* a local minima of $f(\mathbf{x})$ satisfying the constraints $\mathbf{h} \in C^2(\mathbb{R}^n, \mathbb{R}^m)$, i.e. $\mathbf{h}(\mathbf{x}^*) = \mathbf{0}$. If $\nabla \mathbf{h}(\mathbf{x}^*)$ is full rank then there exists m scalars λ_k such that

$$\nabla_{\mathbf{x}} \mathcal{L}(\mathbf{x}^*, \boldsymbol{\lambda}) = \nabla f(\mathbf{x}^*) - \sum_{k=1}^m \lambda_k \nabla h_k(\mathbf{x}^*) = \mathbf{0}^T \quad (\text{A})$$

moreover, for all $\mathbf{z} \in \mathbb{R}^n$ which satisfy $\nabla \mathbf{h}(\mathbf{x}^*) \mathbf{z} = \mathbf{0}$ it follows

$$\mathbf{z}^T \nabla_{\mathbf{x}}^2 \mathcal{L}(\mathbf{x}^*, \boldsymbol{\lambda}) \mathbf{z} = \mathbf{z}^T \left(\nabla^2 f(\mathbf{x}^*) - \sum_{k=1}^m \lambda_k \nabla^2 h_k(\mathbf{x}^*) \right) \mathbf{z} \geq 0 \quad (\text{B})$$

in other words the matrix $\nabla_{\mathbf{x}}^2 (f(\mathbf{x}^*) - \boldsymbol{\lambda} \cdot \mathbf{h}(\mathbf{x}^*))$ is semi-SPD in the Kernel of $\nabla \mathbf{h}(\mathbf{x}^*)$.

Proof. Let \mathbf{x}^* a local minima, then there exists $\varepsilon > 0$ such that

$$f(\mathbf{x}^*) \leq f(\mathbf{x}), \quad \text{for all } \mathbf{x} \in B \text{ with } \mathbf{h}(\mathbf{x}) = \mathbf{0}, \quad (1)$$

where $B = \{\mathbf{x} \mid \|\mathbf{x} - \mathbf{x}^*\| \leq \varepsilon\}$. Consider thus, the functions sequence

$$f_k(\mathbf{x}) = f(\mathbf{x}) + k\|\mathbf{h}(\mathbf{x})\|^2 + \alpha\|\mathbf{x} - \mathbf{x}^*\|^2, \quad \alpha > 0 \quad (2)$$

with the corresponding sequence of (unconstrained) local minima in B :

$$\mathbf{x}_k = \underset{\mathbf{x} \in B}{\operatorname{argmin}} f_k(\mathbf{x}).$$

The sequence \mathbf{x}_k is contained in the compact ball B and from compactness there exists a converging sub-sequence $\mathbf{x}_{k_j} \rightarrow \bar{\mathbf{x}} \in B$. The rest of the proof to verify that $\bar{\mathbf{x}} = \mathbf{x}^*$ and it a minimum.

Step 1: $\mathbf{h}(\bar{\mathbf{x}}) = \mathbf{0}$. Notice that the sequence x_k satisfy $f_k(\mathbf{x}_k) \leq f(\mathbf{x}^*)$, in fact

$$f_k(\mathbf{x}_k) \leq f_k(\mathbf{x}^*) = f(\mathbf{x}^*) + k\|\mathbf{h}(\mathbf{x}^*)\|^2 + \alpha\|\mathbf{x}^* - \mathbf{x}^*\|^2 = f(\mathbf{x}^*).$$

and by definition (2) we have

$$\begin{aligned} k_j \|\mathbf{h}(\mathbf{x}_{k_j})\|^2 + \alpha \|\mathbf{x}_{k_j} - \mathbf{x}^*\|^2 &\leq f(\mathbf{x}^*) - f(\mathbf{x}_{k_j}) \\ &\leq f(\mathbf{x}^*) - \min_{\mathbf{x} \in B} f(\mathbf{x}) = C < +\infty \end{aligned} \quad (3)$$

so that

$$\lim_{j \rightarrow \infty} \|\mathbf{h}(\mathbf{x}_{k_j})\| = 0 \quad \Rightarrow \quad \left\| \mathbf{h} \left(\lim_{j \rightarrow \infty} \mathbf{x}_{k_j} \right) \right\| = \|\mathbf{h}(\bar{\mathbf{x}})\| = 0 \quad \Rightarrow \quad \mathbf{h}(\bar{\mathbf{x}}) = \mathbf{0}.$$

Step 2: $\bar{\mathbf{x}} = \mathbf{x}^*$. From (3)

$$\alpha \|\mathbf{x}_{k_j} - \mathbf{x}^*\|^2 \leq f(\mathbf{x}^*) - f(\mathbf{x}_{k_j}) - k_j \|\mathbf{h}(\mathbf{x}_{k_j})\|^2 \leq f(\mathbf{x}^*) - f(\mathbf{x}_{k_j})$$

and taking the limit

$$\alpha \left\| \lim_{j \rightarrow \infty} \mathbf{x}_{k_j} - \mathbf{x}^* \right\|^2 \leq \alpha \|\bar{\mathbf{x}} - \mathbf{x}^*\|^2 \leq f(\mathbf{x}^*) - \lim_{j \rightarrow \infty} f(\mathbf{x}_{k_j}) \leq f(\mathbf{x}^*) - f(\bar{\mathbf{x}})$$

From $\|\mathbf{h}(\bar{\mathbf{x}})\| = 0$ it follows that from (1) that $f(\mathbf{x}^*) \leq f(\bar{\mathbf{x}})$ and

$$\alpha \|\bar{\mathbf{x}} - \mathbf{x}^*\|^2 \leq f(\mathbf{x}^*) - f(\bar{\mathbf{x}}) \leq 0$$

and, thus $\bar{\mathbf{x}} = \mathbf{x}^*$.

Step 3: Build multiplier. Cause \mathbf{x}_{k_j} are *unconstrained local minima* for $f_{k_j}(\mathbf{x})$ it follows

$$\nabla f_{k_j}(\mathbf{x}_{k_j}) = \nabla f(\mathbf{x}_{k_j}) + k_j \nabla \|\mathbf{h}(\mathbf{x}_{k_j})\|^2 + \alpha \nabla \|\mathbf{x}_{k_j} - \mathbf{x}^*\|^2 = \mathbf{0}$$

remembering that

$$\begin{aligned} \nabla \|\mathbf{x}\|^2 &= \nabla(\mathbf{x} \cdot \mathbf{x}) = 2\mathbf{x}^T, \\ \nabla \|\mathbf{h}(\mathbf{x})\|^2 &= \nabla(\mathbf{h}(\mathbf{x}) \cdot \mathbf{h}(\mathbf{x})) = 2\mathbf{h}(\mathbf{x})^T \nabla \mathbf{h}(\mathbf{x}), \end{aligned}$$

it follows (doing transposition)

$$\nabla f(\mathbf{x}_{k_j})^T + 2k_j \nabla \mathbf{h}(\mathbf{x}_{k_j})^T \mathbf{h}(\mathbf{x}_{k_j}) + 2\alpha(\mathbf{x}_{k_j} - \mathbf{x}^*) = \mathbf{0}. \quad (4)$$

Left multiplying by $\nabla \mathbf{h}(\mathbf{x}_{k_j})$

$$\nabla \mathbf{h}(\mathbf{x}_{k_j}) \left[\nabla f(\mathbf{x}_{k_j})^T + 2\alpha(\mathbf{x}_{k_j} - \mathbf{x}^*) \right] + 2k_j \nabla \mathbf{h}(\mathbf{x}_{k_j}) \nabla \mathbf{h}(\mathbf{x}_{k_j})^T \mathbf{h}(\mathbf{x}_{k_j}) = \mathbf{0}$$

Cause $\nabla \mathbf{h}(\mathbf{x}^*) \in \mathbb{R}^{m \times n}$ is full rank for j large by continuity $\nabla \mathbf{h}(\mathbf{x}_{k_j})$ is full rank and thus $\nabla \mathbf{h}(\mathbf{x}_{k_j}) \nabla \mathbf{h}(\mathbf{x}_{k_j})^T \in \mathbb{R}^{m \times m}$ are nonsingular, thus

$$2k_j \mathbf{h}(\mathbf{x}_{k_j}) = - \left(\nabla \mathbf{h}(\mathbf{x}_{k_j}) \nabla \mathbf{h}(\mathbf{x}_{k_j})^T \right)^{-1} \nabla \mathbf{h}(\mathbf{x}_{k_j}) \left[\nabla f(\mathbf{x}_{k_j})^T + 2\alpha(\mathbf{x}_{k_j} - \mathbf{x}^*) \right]$$

taking the limit for $j \rightarrow \infty$

$$\lim_{j \rightarrow \infty} 2k_j \mathbf{h}(\mathbf{x}_{k_j}) = - \left(\nabla \mathbf{h}(\mathbf{x}^*) \nabla \mathbf{h}(\mathbf{x}^*)^T \right)^{-1} \nabla \mathbf{h}(\mathbf{x}^*) \nabla f(\mathbf{x}^*)^T = -\lambda \quad (5)$$

and taking the limit of (4) with (5) we have $\nabla f(\mathbf{x}^*)^T - \nabla \mathbf{h}(\mathbf{x}^*)^T \lambda = \mathbf{0}$.

Step 4: Build a special sequence of z_j . We needs a sequence $z_j \rightarrow z$ such that $\nabla \mathbf{h}(\mathbf{x}_{k_j}) z_j = \mathbf{0}$ for all j . The sequence z_j is built as the projection of z into the Kernel of $\nabla \mathbf{h}(\mathbf{x}_{k_j})$, i.e.

$$z_j = z - \nabla \mathbf{h}(\mathbf{x}_{k_j})^T \left[\nabla \mathbf{h}(\mathbf{x}_{k_j}) \nabla \mathbf{h}(\mathbf{x}_{k_j})^T \right]^{-1} \nabla \mathbf{h}(\mathbf{x}_{k_j}) z$$

infact

$$\begin{aligned} \nabla \mathbf{h}(\mathbf{x}_{k_j}) z_j &= \nabla \mathbf{h}(\mathbf{x}_{k_j}) z - \nabla \mathbf{h}(\mathbf{x}_{k_j}) \nabla \mathbf{h}(\mathbf{x}_{k_j})^T \left[\nabla \mathbf{h}(\mathbf{x}_{k_j}) \nabla \mathbf{h}(\mathbf{x}_{k_j})^T \right]^{-1} \nabla \mathbf{h}(\mathbf{x}_{k_j}) z \\ &= \nabla \mathbf{h}(\mathbf{x}_{k_j}) z - \nabla \mathbf{h}(\mathbf{x}_{k_j}) z = \mathbf{0} \end{aligned}$$

consider now the limit

$$\begin{aligned} \lim_{j \rightarrow \infty} z_j &= z - \lim_{j \rightarrow \infty} \nabla \mathbf{h}(\mathbf{x}_{k_j})^T \left[\nabla \mathbf{h}(\mathbf{x}_{k_j}) \nabla \mathbf{h}(\mathbf{x}_{k_j})^T \right]^{-1} \nabla \mathbf{h}(\mathbf{x}_{k_j}) z \\ &= z - \nabla \mathbf{h}(\mathbf{x}^*)^T \left[\nabla \mathbf{h}(\mathbf{x}^*) \nabla \mathbf{h}(\mathbf{x}^*)^T \right]^{-1} \nabla \mathbf{h}(\mathbf{x}^*) z \end{aligned}$$

and thus, if z is in the kernel of $\nabla \mathbf{h}(\mathbf{x}^*)$, i.e. $\nabla \mathbf{h}(\mathbf{x}^*)z = \mathbf{0}$ we have

$$\nabla \mathbf{h}(\mathbf{x}_{k_j})z_j = \mathbf{0} \quad \text{with} \quad \lim_{j \rightarrow \infty} z_j = z.$$

Step 5: Necessary conditions. Cause \mathbf{x}_{k_j} are *unconstrained local minima* for $f_{k_j}(\mathbf{x})$ it follows that matrices $\nabla^2 f_{k_j}(\mathbf{x}_{k_j})$ are semi positive defined, i.e.

$$z^T \nabla^2 f_{k_j}(\mathbf{x}_{k_j})z \geq 0, \quad \forall z \in \mathbb{R}^n$$

moreover

$$\begin{aligned} \nabla^2 f_{k_j}(\mathbf{x}_{k_j}) &= \nabla^2 f(\mathbf{x}_{k_j}) + k \nabla^2 \left\| \mathbf{h}(\mathbf{x}_{k_j}) \right\|^2 + 2\alpha \nabla(\mathbf{x}_{k_j} - \mathbf{x}^*) \\ &= \nabla^2 f(\mathbf{x}_{k_j})^T + k \nabla^2 \sum_{i=1}^m h_i(\mathbf{x}_{k_j})^2 + 2\alpha \mathbf{I} \end{aligned} \quad (6)$$

using the identity

$$\nabla^2 h(\mathbf{x})^2 = \nabla(2h(\mathbf{x})\nabla h(\mathbf{x})^T) = 2\nabla h(\mathbf{x})^T \nabla h(\mathbf{x}) + 2h(\mathbf{x})\nabla^2 h(\mathbf{x})$$

in (8)

$$\nabla^2 f_{k_j}(\mathbf{x}_{k_j}) = \nabla^2 f(\mathbf{x}_{k_j}) + 2k_j \sum_{i=1}^m \nabla h_i(\mathbf{x}_{k_j})^T \nabla h_i(\mathbf{x}_{k_j}) + 2k_j \sum_{i=1}^m h_i(\mathbf{x}_{k_j}) \nabla^2 h_i(\mathbf{x}_{k_j}) + 2\alpha \mathbf{I}$$

Let $z \in \mathbb{R}^n$ then $0 \leq z^T \nabla^2 f_{k_j}(\mathbf{x}_{k_j})z$, i.e.

$$0 \leq z^T \nabla^2 f(\mathbf{x}_{k_j})z + \sum_{i=1}^m (2k_j h_i(\mathbf{x}_{k_j})) z^T \nabla^2 h_i(\mathbf{x}_{k_j})z + 2k_j \left\| \nabla \mathbf{h}(\mathbf{x}_{k_j})z \right\|^2 + 2\alpha \|z\|^2$$

Inequality is true for all $z \in \mathbb{R}^n$ and thus for any z in the kernel of $\nabla \mathbf{h}(\mathbf{x}^*)$. Choosing z in the kernel of $\nabla \mathbf{h}(\mathbf{x}^*)$ from previous step the sequence z_j satisfy

$$0 \leq z_j^T \nabla^2 f(\mathbf{x}_{k_j})z_j + \sum_{i=1}^m (2k_j h_i(\mathbf{x}_{k_j})) z_j^T \nabla^2 h_i(\mathbf{x}_{k_j})z_j + 2\alpha \|z_j\|^2$$

and taking the limit $j \rightarrow \infty$ with (5)

$$0 \leq z^T \nabla^2 f(\mathbf{x}^*)z + \sum_{i=1}^m \lambda_i z^T \nabla^2 h_i(\mathbf{x}^*)z + 2\alpha \|z\|^2$$

cause $\alpha > 0$ can be chosen arbitrarily it follows

$$0 \leq z^T \nabla^2 f(\mathbf{x}^*) z - \sum_{i=1}^m \lambda_i [z^T \nabla^2 h_i(\mathbf{x}^*) z]$$

which is the relation to be proved. \square

Inequality constraints

It is possible to adapt theorem 1 for inequality constraints. Consider the NLP problem

$$\begin{aligned} \text{minimize:} \quad & f(\mathbf{x}) \\ \text{subject to:} \quad & h_i(\mathbf{x}) = 0 \quad i = 1, 2, \dots, m \\ & g_i(\mathbf{x}) \geq 0 \quad i = 1, 2, \dots, p \end{aligned}$$

introducing the *slack* variables $e_i, i = 1, 2, \dots, p$ and $\mathbf{y}^T = (\mathbf{x}^T, e^T)$ the new problem

$$\begin{aligned} \text{minimize:} \quad & \mathbf{f}(\mathbf{y}) = f(\mathbf{x}) \\ \text{subject to:} \quad & h_i(\mathbf{y}) = h_i(\mathbf{x}) = 0 \quad i = 1, 2, \dots, m \\ & h_{i+m}(\mathbf{y}) = g_i(\mathbf{x}) - e_i^2 = 0 \quad i = 1, 2, \dots, p \end{aligned}$$

with the Lagrangian function:

$$\mathcal{L}(\mathbf{x}, \mathbf{e}, \boldsymbol{\lambda}, \boldsymbol{\mu}) = f(\mathbf{x}) - \sum_{k=1}^m \lambda_k h_k(\mathbf{x}) - \sum_{k=1}^p \mu_k (g_k(\mathbf{x}) - e_k^2)$$

The first order condition becomes

$$\begin{aligned} \nabla_{\mathbf{x}} \mathcal{L}(\mathbf{x}^*, \mathbf{e}, \boldsymbol{\lambda}, \boldsymbol{\mu}) &= \nabla f(\mathbf{x}^*) - \sum_{k=1}^m \lambda_k \nabla h_k(\mathbf{x}^*) - \sum_{k=1}^p \mu_k \nabla g_k(\mathbf{x}^*) = \mathbf{0}^T, \\ \nabla_{\mathbf{e}} \mathcal{L}(\mathbf{x}^*, \mathbf{e}, \boldsymbol{\lambda}, \boldsymbol{\mu}) &= 2(\mu_1 e_1, \dots, \mu_p e_p) = \mathbf{0}^T, \\ h_k(\mathbf{x}^*) &= 0, \\ g_k(\mathbf{x}^*) &= e_k^2 \geq 0, \end{aligned}$$

and second order condition become $z^T \nabla_{(\mathbf{x}, \mathbf{e})}^2 \mathcal{L}(\mathbf{x}^*, \mathbf{e}, \boldsymbol{\lambda}, \boldsymbol{\mu}) z \geq 0$ for z in the kernel of matrix

$$\begin{pmatrix} \nabla_{\mathbf{x}} \mathbf{h}(\mathbf{x}^*) & \mathbf{0} \\ \nabla_{\mathbf{x}} \mathbf{g}(\mathbf{x}^*) & 2 \text{diag}(e_1, \dots, e_p) \end{pmatrix} \quad (7)$$

where

$$\nabla_{(x,e)}^2 \mathcal{L}(\mathbf{x}^*, e, \lambda, \mu) \mathbf{z} = \begin{pmatrix} \nabla_x^2 \mathcal{L}(\mathbf{x}^*, e, \lambda, \mu) & \mathbf{0} \\ \mathbf{0} & \nabla_e^2 \mathcal{L}(\mathbf{x}^*, e, \lambda, \mu) \end{pmatrix} \quad (8)$$

and $\nabla_x \nabla_e^T \mathcal{L}(\mathbf{x}^*, e, \lambda, \mu) = \mathbf{0}$, moreover

$$\nabla_x^2 \mathcal{L}(\mathbf{x}^*, e, \lambda, \mu) = \nabla^2 f(\mathbf{x}^*) - \sum_{k=1}^m \lambda_k \nabla^2 h_k(\mathbf{x}^*) - \sum_{k=1}^p \mu_k \nabla^2 g_k(\mathbf{x}^*),$$

$$\nabla_e^2 \mathcal{L}(\mathbf{x}^*, e, \lambda, \mu) = 2 \operatorname{diag}(\mu_1, \mu_2, \dots, \mu_p).$$

Notice that $\mu_k e_k = 0$ is equivalent of $\mu_k e_k^2 = 0$ and thus $\mu_k g_k(\mathbf{x}^*) = 0$. So that when $g_k(\mathbf{x}^*) > 0$ then $\mu_k = 0$. Up to a reordering we split $\mathbf{g}(\mathbf{x}) = \begin{pmatrix} \mathbf{g}^{(1)}(\mathbf{x}) \\ \mathbf{g}^{(2)}(\mathbf{x}) \end{pmatrix}$ where

$$\begin{aligned} g_k(\mathbf{x}^*) &= e_k^2 = 0, & k &= 1, 2, \dots, r \\ g_k(\mathbf{x}^*) &= e_k^2 > 0, & k &= r+1, r+2, \dots, p \end{aligned}$$

and thus (7) becomes

$$\begin{pmatrix} \nabla_x \mathbf{h}(\mathbf{x}^*) & \mathbf{0} & \mathbf{0} \\ \nabla_x \mathbf{g}^{(1)}(\mathbf{x}^*) & \mathbf{0} & \mathbf{0} \\ \nabla_x \mathbf{g}^{(2)}(\mathbf{x}^*) & \mathbf{0} & \mathbf{E} \end{pmatrix}, \quad 2 \operatorname{diag}(e_{k+1}, \dots, e_p) = \mathbf{E}. \quad (9)$$

and

$$\nabla_e^2 \mathcal{L}(\mathbf{x}^*, e, \lambda, \mu) = \begin{pmatrix} \mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, \quad \mathbf{M} = 2 \operatorname{diag}(\mu_1, \mu_2, \dots, \mu_r) \quad (10)$$

The group of constraints $\mathbf{g}^{(1)}(\mathbf{x}^*)$ that are zeros are the active constraints. The kernel of (9) can be written as

$$\mathcal{K} = \begin{pmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \\ -\mathbf{E}^{-1} \nabla_x \mathbf{g}^{(2)}(\mathbf{x}^*) \mathbf{K} & \mathbf{0} \end{pmatrix}, \quad \mathbf{K} \text{ is the kernel of } \begin{pmatrix} \nabla_x \mathbf{h}(\mathbf{x}^*) \\ \nabla_x \mathbf{g}^{(1)}(\mathbf{x}^*) \end{pmatrix} \quad (11)$$

where \mathbf{K} is the kernel of the matrix

$$\begin{pmatrix} \nabla_x \mathbf{h}(\mathbf{x}^*) \\ \nabla_x \mathbf{g}^{(1)}(\mathbf{x}^*) \end{pmatrix}$$

thus z can be written as $\mathcal{K}d$ and thus second order condition $z^T \nabla_{(x,e)}^2 \mathcal{L}(x^*, e, \lambda, \mu) z \geq 0$ become

$$0 \leq d^T \left[\mathcal{K}^T \nabla_{(x,e)}^2 \mathcal{L}(x^*, e, \lambda, \mu) \mathcal{K} \right] d, \quad d \in \mathbb{R}^s$$

and using (11) with (8) and (10)

$$\begin{aligned} \left[\mathcal{K}^T \nabla_{(x,e)}^2 \mathcal{L}(x^*, e, \lambda, \mu) \mathcal{K} \right] &= \mathcal{K}^T \begin{pmatrix} \nabla_x^2 \mathcal{L}(x^*, e, \lambda, \mu) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & M & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \mathcal{K}, \\ &= \begin{pmatrix} \mathbf{K}^T \nabla_x^2 \mathcal{L}(x^*, e, \lambda, \mu) \mathbf{K} & \mathbf{0} \\ \mathbf{0} & M \end{pmatrix} \end{aligned}$$

Using the solution algorithm of the equality constrained problem we have

- Necessary condition: the matrices

$$\mathbf{K}^T \nabla_x^2 \mathcal{L}(x^*, e, \lambda, \mu) \mathbf{K}, \quad \text{and} \quad M$$

must be semi-positive defined. This imply that $\mu_k \geq 0$ for $k = 1, 2, \dots, p$

- Sufficient condition: the matrices

$$\mathbf{K}^T \nabla_x^2 \mathcal{L}(x^*, e, \lambda, \mu) \mathbf{K}, \quad \text{and} \quad M$$

must be positive defined. This imply that $\mu_k > 0$ for the active constraints.

Constrained minima, NLP problem

Consider the constrained minimization problem

$$\begin{aligned}
 &\text{minimize:} && f(\mathbf{x}) \\
 &\text{subject to:} && h_i(\mathbf{x}) = 0 \quad i = 1, 2, \dots, m \\
 &&& g_i(\mathbf{x}) \geq 0 \quad i = 1, 2, \dots, p
 \end{aligned} \tag{12}$$

Solution algorithm

- Compute the Lagrangian function:

$$\mathcal{L}(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}) = f(\mathbf{x}) - \sum_{k=1}^m \lambda_k h_k(\mathbf{x}) - \sum_{k=1}^p \mu_k g_k(\mathbf{x})$$

- Solve the nonlinear system

$$\begin{aligned}
 \nabla_{\mathbf{x}} \mathcal{L}(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}) &= \mathbf{0}^T \\
 h_k(\mathbf{x}) &= 0 \quad k = 1, 2, \dots, m \\
 \mu_k g_k(\mathbf{x}) &= 0 \quad k = 1, 2, \dots, p
 \end{aligned}$$

keep only the solutions with $\mu_k^* \geq 0$ and $g_k(\mathbf{x}^*) \geq 0$.

- For each solution points $(\mathbf{x}^*, \boldsymbol{\lambda}^*, \boldsymbol{\mu}^*)$ compute $\nabla \mathbf{h}(\mathbf{x}^*)$ with $\nabla g_k(\mathbf{x}^*)$ where $g_k(\mathbf{x}^*) = 0$ are the active constraints with $\mu_k > 0$ and check they are linearly independent.
- Compute matrix \mathbf{K} the kernel of $\nabla \mathbf{h}(\mathbf{x}^*)$ with $\nabla g_k(\mathbf{x}^*)$ where $g_k(\mathbf{x}^*) = 0$ are the active constraints with $\mu_k > 0$.
- Compute the reduce Hessian

$$\mathbf{H} = \mathbf{K}^T \nabla_{\mathbf{x}}^2 \mathcal{L}(\mathbf{x}^*, \boldsymbol{\lambda}^*) \mathbf{K},$$

- Necessary condition: \mathbf{H} is semi-positive definite.
- Sufficient condition: \mathbf{H} is positive definite and $\mu_k > 0$ for all the active constraints.

Definition 1 *The set*

$$\mathcal{F} = \{\mathbf{x} \in \mathbb{R}^n \mid h_k(\mathbf{x}) = 0, \quad k = 1, 2, \dots, m, \quad g_k(\mathbf{x}) \geq 0, \quad k = 1, 2, \dots, p, \}$$

is called the feasible region or set of feasible points.

Definition 2 (Active set) The set $\mathcal{A}(\mathbf{x})$ defined as

$$\mathcal{A}(\mathbf{x}) = \{k \mid g_k(\mathbf{x}) = 0\}$$

is the set of active (unilateral) constraints.

Constrained minima general theorem and KKT

The following theorem (see [1]) give the necessary conditions for constrained minima. Notice that no condition on the constraints are necessary.

Theorem 2 (Fritz John) If the functions $f(\mathbf{x}), g_1(\mathbf{x}), \dots, g_p(\mathbf{x})$, are differentiable, then a necessary condition that \mathbf{x}^* be a local minimum to problem:

$$\begin{aligned} \text{minimize:} & \quad f(\mathbf{x}) \\ \text{subject to:} & \quad g_i(\mathbf{x}) \geq 0 \quad i = 1, 2, \dots, p \end{aligned}$$

is that there exist scalars $\mu_0^*, \mu_1^*, \mu_p^*$, (not all zero) such that the following inequalities and equalities are satisfied:

$$\begin{aligned} \nabla_x \mathcal{L}(\mathbf{x}^*, \boldsymbol{\mu}^*) &= \mathbf{0}^T \\ \mu_k^* g_k(\mathbf{x}^*) &= 0, \quad k = 1, 2, \dots, p; \\ \mu_k^* &\geq 0, \quad k = 0, 1, 2, \dots, p; \end{aligned}$$

where

$$\mathcal{L}(\mathbf{x}, \boldsymbol{\mu}) = f(\mathbf{x}) - \sum_{k=1}^p \mu_k g_k(\mathbf{x})$$

In [2] Kuhn and Tucker showed that if a condition, called the *first order constraint qualification*, holds at $\mathbf{x}^*, \boldsymbol{\lambda}^*$ then λ_0 can be taken equal to 1.

Definition 3 (Constraints qualification LI) Let be the unilateral and bilateral constraints $\mathbf{g}(\mathbf{x})$ and $\mathbf{h}(\mathbf{x})$, the point \mathbf{x}^* is admissible if

$$g_k(\mathbf{x}^*) \geq 0, \quad h_k(\mathbf{x}^*) = 0.$$

The constraints $\mathbf{g}(\mathbf{x})$ and $\mathbf{h}(\mathbf{x})$ are qualified at \mathbf{x}^* if the point \mathbf{x}^* is admissible and the vectors

$$\{\nabla g_k(\mathbf{x}^*) : k \in \mathcal{A}(\mathbf{x}^*)\} \cup \{\nabla h_1(\mathbf{x}^*), \nabla h_2(\mathbf{x}^*), \dots, \nabla h_m(\mathbf{x}^*)\}$$

are linearly independent.

Definition 4 (Constraint qualification (Mangasarian-Fromovitz)) The constraints $\mathbf{g}(\mathbf{x})$ and $\mathbf{h}(\mathbf{x})$ are qualified at \mathbf{x}^* if the point \mathbf{x}^* is admissible and **does not exist** a linear combination

$$\sum_{k \in \mathcal{A}(\mathbf{x}^*)}^m \alpha_k \nabla g_k(\mathbf{x}^*) + \sum_{k=1}^m \beta_k \nabla h_k(\mathbf{x}^*) = \mathbf{0}$$

with $\alpha_k \geq 0$ for $k \in \mathcal{A}(\mathbf{x}^*)$ and α_k with β_k not all 0. In other words, there not exists a non trivial linear combination of the null vector such that $\alpha_k \geq 0$ for $k \in \mathcal{A}(\mathbf{x}^*)$.

The next theorems are taken from [3].

Theorem 3 (First order necessary conditions) Let $f \in C^1(\mathbb{R}^n)$ and the constraints $\mathbf{g} \in C^1(\mathbb{R}^n, \mathbb{R}^p)$ and $\mathbf{h} \in C^1(\mathbb{R}^n, \mathbb{R}^m)$. Suppose that \mathbf{x}^* is a local minima of (12) and that the constraints qualification LI holds at \mathbf{x}^* . Then there are Lagrange multiplier vectors $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ such that the following conditions are satisfied at $(\mathbf{x}^*, \boldsymbol{\lambda}, \boldsymbol{\mu})$

$$\begin{aligned} \nabla_x \mathcal{L}(\mathbf{x}^*, \boldsymbol{\lambda}^*, \boldsymbol{\mu}^*) &= \mathbf{0}^T \\ h_k(\mathbf{x}^*) &= 0, \quad k = 1, 2, \dots, m; \\ \mu_k^* g_k(\mathbf{x}^*) &= 0, \quad k = 1, 2, \dots, p; \\ \mu_k^* &\geq 0, \quad k = 1, 2, \dots, p; \end{aligned}$$

where

$$\mathcal{L}(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}) = f(\mathbf{x}) - \sum_{k=1}^m \lambda_k h_k(\mathbf{x}) - \sum_{k=1}^p \mu_k g_k(\mathbf{x})$$

Theorem 4 (Second order necessary conditions) Let $f \in C^2(\mathbb{R}^n)$ and the constraints $\mathbf{g} \in C^2(\mathbb{R}^n, \mathbb{R}^p)$ and $\mathbf{h} \in C^2(\mathbb{R}^n, \mathbb{R}^m)$. Let \mathbf{x}^* satisfying the First order necessary conditions, a **necessary** condition for \mathbf{x}^* be a local minima is that the $m + p$ scalars (Lagrange Multiplier) of the first order necessary condition satisfy:

$$\mathbf{d}^T \nabla_x^2 \mathcal{L}(\mathbf{x}^*, \boldsymbol{\lambda}^*, \boldsymbol{\mu}^*) \mathbf{d} \geq 0$$

for all \mathbf{d} such that

$$\begin{aligned} \nabla h_k(\mathbf{x}^*) \mathbf{d} &= 0, \quad k = 1, 2, \dots, m \\ \nabla g_k(\mathbf{x}^*) \mathbf{d} &= 0, \quad \text{if } k \in \mathcal{A}(\mathbf{x}^*) \text{ and } \mu_k > 0 \\ \nabla g_k(\mathbf{x}^*) \mathbf{d} &\geq 0, \quad \text{if } k \in \mathcal{A}(\mathbf{x}^*) \text{ and } \mu_k = 0 \end{aligned}$$

Remark 1 *The conditions*

$$\nabla g_k(\mathbf{x}^*)\mathbf{d} = 0, \quad \text{if } k \in \mathcal{A}(\mathbf{x}^*) \text{ and } \mu_k > 0$$

$$\nabla g_k(\mathbf{x}^*)\mathbf{d} \geq 0, \quad \text{if } k \in \mathcal{A}(\mathbf{x}^*) \text{ and } \mu_k = 0$$

restrict the space of direction to be considered. If changed with

$$\nabla g_k(\mathbf{x}^*)\mathbf{d} = 0, \quad \text{if } k \in \mathcal{A}(\mathbf{x}^*)$$

theorems 4 is still valid cause necessary condition is tested in a smaller set.

Theorem 5 (Second order sufficient conditions) Let $f \in C^2(\mathbb{R}^n)$ and the constraints $g \in C^2(\mathbb{R}^n, \mathbb{R}^p)$ and $h \in C^2(\mathbb{R}^n, \mathbb{R}^m)$. Let \mathbf{x}^* satisfying the First order necessary conditions, a **sufficient** condition for \mathbf{x}^* be a local minima is that the $m+p$ scalars (Lagrange Multiplier) of the first order necessary condition satisfy:

$$\mathbf{d}^T \nabla_x^2 \mathcal{L}(\mathbf{x}^*, \boldsymbol{\lambda}^*, \boldsymbol{\mu}^*) \mathbf{d} > 0$$

for all $\mathbf{d} \neq \mathbf{0}$ such that

$$\nabla h_k(\mathbf{x}^*)\mathbf{d} = 0, \quad k = 1, 2, \dots, m$$

$$\nabla g_k(\mathbf{x}^*)\mathbf{d} = 0, \quad \text{if } k \in \mathcal{A}(\mathbf{x}^*) \text{ and } \mu_k > 0$$

$$\nabla g_k(\mathbf{x}^*)\mathbf{d} \geq 0, \quad \text{if } k \in \mathcal{A}(\mathbf{x}^*) \text{ and } \mu_k = 0$$

Remark 2 *The condition*

$$\nabla g_k(\mathbf{x}^*)\mathbf{d} \geq 0, \quad \text{if } k \in \mathcal{A}(\mathbf{x}^*) \text{ and } \mu_k = 0$$

restrict the space of direction to be considered. If omitted the theorems 5 is still valid cause sufficient condition is tested in a larger set.

References

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