The Z transform
(Computational Methods for Mechatronics [140466])

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AA 2014/2015

Outline

Definition

- Z is applied to causal discrete signal.
- Discrete signal is denoted using various notations:

$$f_n : n = 0, 1, 2, \ldots$$

$$f[n] : n = 0, 1, 2, \ldots$$

$$f(n) : n = 0, 1, 2, \ldots$$

- Z transform is defined as:

$$\mathcal{Z}\{f_n\}(z) = \sum_{n=0}^{\infty} f_n z^{-n}$$

- A lighter notation for Z-transform is:

$$\mathcal{Z}\{f_n\}(z) \equiv \widetilde{f}(z)$$

The Z transform

- Usefulness: analysis of digital signal; transform

  Difference equations $\Rightarrow$ Algebraic equations

- Analogy with logarithm:

$$a \rightarrow \log a$$

$$a \cdot b \rightarrow \log a + \log b$$

thus, logarithm transform product into addition which are easier to manage.

Linearity of Z-transform

Let f_n and g_n two discrete signals and α and β two scalars

$$\begin{aligned}
\mathcal{Z}\left\{\alpha f_n + \beta g_n\right\}(z) &= \sum_{n=0}^{\infty} (\alpha f_n + \beta g_n) z^{-n} \\
&= \alpha \sum_{n=0}^{\infty} f_n z^{-n} + \beta \sum_{n=0}^{\infty} g_n z^{-n} \\
&= \alpha \mathcal{Z}\left\{f_n\right\}(z) + \beta \mathcal{Z}\left\{g_n\right\}(z)
\end{aligned}$$

Unitary impulse

- Unitary impulse is defined as

$$\delta_n = \begin{cases} 1 & \text{se } n = 0, \\ 0 & \text{se } n > 0, \end{cases}$$

- Its Z-transform is

$$\mathcal{Z}\left\{\delta_n\right\}(z) = \sum_{n=0}^{\infty} \delta_n z^{-n}$$
$$= 1$$

Discrete Heaviside or step signal

- Discrete Heaviside is defined as

$$\mathbf{1} = \{1\}_{n=0}^{\infty}$$

- Its Z-transform is

$$\begin{aligned}\mathcal{Z}\{\delta_n\}(z) &= \sum_{n=0}^{\infty} \mathbf{1}_n z^{-n} \\ &= \sum_{n=0}^{\infty} z^{-n} \\ &= \frac{1}{1 - z^{-1}} = \frac{z}{z - 1}\end{aligned}$$

Esponential

- Z-transform of exponential a^n

$$\mathcal{Z}\{a^n\}(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n$$

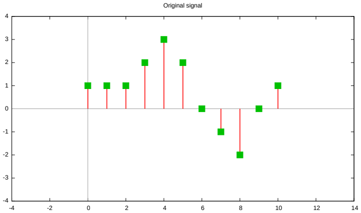
$$= \frac{1}{1 - a/z} = \frac{z}{z - a}$$

- Z-transform of exponential a^n multiply by a signal

$$\mathcal{Z}\{a^n f_n\}(z) = \sum_{n=0}^{\infty} f_n a^n z^{-n} = \sum_{n=0}^{\infty} f_n \left(\frac{z}{a}\right)^{-n}$$

$$= \mathcal{Z}\{f_n\}\left(\frac{z}{a}\right)$$

Shift (time advance) (1/2)



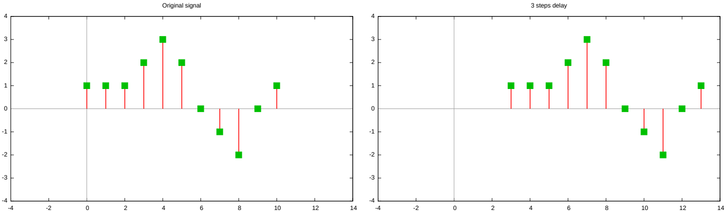
Shift (time advance) (2/2)

Z-Transform of shifted signal f_{n+k} with integer $k > 0$

$$\begin{aligned}
\mathcal{Z}\left\{f_{n+k}\right\}(z) &= \sum_{n=0}^{\infty} f_{n+k} z^{-n} \\
&= z^k \sum_{n=0}^{\infty} f_{n+k} z^{-(n+k)} \\
&= z^k \sum_{n=0}^{\infty} f_n z^{-n} - z^k \sum_{n=0}^{k-1} f_n z^{-n} \\
&= z^k \left(\mathcal{Z}\left\{f_n\right\}(z) - \sum_{n=0}^{k-1} f_n z^{-n} \right)
\end{aligned}$$

Observation: Analogously with Laplace Transform there are *initial conditions* to manage.

Shift (time delay) (1/2)



Shift (time delay) (2/2)

Z-Transform of shifted signal f_{n+k} with integer $k > 0$

$$\begin{aligned}
\mathcal{Z}\left\{f_{n-k}\right\}(z) &= \sum_{n=0}^{\infty} f_{n-k} z^{-n} \\
&= z^{-k} \sum_{n=0}^{\infty} f_{n-k} z^{-(n-k)} \\
&= z^{-k} \sum_{n=0}^{\infty} f_n z^{-n} \\
&= z^{-k} \mathcal{Z}\left\{f_n\right\}(z)
\end{aligned}$$

Osservazione: Differently with Laplace Transform there are NO *initial conditions* to manage. Why ?

The signal n_k (1/2)

- The signal n_k is defined as:

$$n_k = n(n-1)(n-2)\cdots(n-k+1)$$

- Particular cases:
  - $n_0 = 1$;
  - $n_1 = n$.
- Observe that

$$\frac{\mathrm{d}^k}{\mathrm{d}w^k} w^n = n(n-1)(n-2)\cdots(n-k+1)w^{n-k}$$

and

$$\mathcal{Z}\left\{n_k\right\}(1/w) = \sum_{n=0}^{\infty} n_k w^n = w^k \frac{\mathrm{d}^k}{\mathrm{d}w^k} \sum_{n=0}^{\infty} w^n$$

The signal n_k (2/2)

Z-transform is:

$$\begin{aligned}
\mathcal{Z}\{n_k\}(1/w) &= w^k \frac{\mathrm{d}^k}{\mathrm{d}w^k}\frac{1}{1-w} &&= w^k \frac{\mathrm{d}^k}{\mathrm{d}w^k}(1-w)^{-1} \\
&= w^k \frac{\mathrm{d}^{k-1}}{\mathrm{d}w^{k-1}}(1-w)^{-2} &&= w^k 2\frac{\mathrm{d}^{k-2}}{\mathrm{d}w^{k-2}}(1-w)^{-3} \\
&= w^k 2\cdot 3\frac{\mathrm{d}^{k-3}}{\mathrm{d}w^{k-3}}(1-w)^{-4} &&= \cdots = \\
&= w^k \frac{k!}{(1-w)^{k+1}}
\end{aligned}$$

substituting $z = 1/w$:

$$\mathcal{Z}\{n_k\}(z) = \frac{z\,k!}{(z-1)^{k+1}}$$

Binomial coefficient signal

- The signal is defined as:

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}$$

- Particular case:
  - $\binom{n}{0} = 1$;
  - $\binom{n}{1} = n$.
- Observe that

$$\binom{n}{k} = \frac{n_k}{k!}$$

and

$$\mathcal{Z}\left\{\binom{n}{k}\right\}(z) = \frac{z}{(z-1)^{k+1}}$$

Z-transform of the convolution (1/2)

- The convolution of two signals f_n and g_n is defined as

$$(f \star g)_n = \sum_{k=0}^{n} f_k g_{n-k}$$

- For convolution Z-transform $(f \star g)(z) = \mathcal{Z}\left\{(f \star g)_n\right\}(z)$ the following property hold:

$$(f \star g)(z) = \widetilde{f}(z)\widetilde{g}(z)$$

Z-transform of the convolution (2/2)

The Z-transform is

$$\begin{aligned}
\mathcal{Z}\left\{(f \star g)_n\right\}(z) &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} f_k g_{n-k} z^{-n} \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \mathbf{1}_{n-k} f_k g_{n-k} z^{-(n-k)} z^{-k} \\
&= \sum_{k=0}^{\infty} f_k z^{-k} \left(\sum_{n=0}^{\infty} \mathbf{1}_{n-k} g_{n-k} z^{-(n-k)} \right) \\
&= \sum_{k=0}^{\infty} f_k z^{-k} \left(\sum_{n=0}^{\infty} g_n z^{-n} \right) \\
&= \widetilde{f}(z) \widetilde{g}(z)
\end{aligned}$$

The signal $\cos \omega n$

Using equality $2\cos\alpha = e^{i\alpha} + e^{-i\alpha}$ where i is the imaginary unit for complex numbers, now:

$$\mathcal{Z}\left\{\cos\omega n\right\}(z) = \sum_{n=0}^{\infty} \cos\omega n\, z^{-n} = \frac{1}{2}\sum_{n=0}^{\infty}(e^{i\omega n} + e^{-i\omega n})z^{-n}$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}(e^{i\omega}z^{-1})^{n} + \frac{1}{2}\sum_{n=0}^{\infty}(e^{-i\omega}z^{-1})^{n}$$

$$= \frac{1}{2}\frac{1}{1 - e^{i\omega}z^{-1}} + \frac{1}{2}\frac{1}{1 - e^{-i\omega}z^{-1}}$$

$$= \frac{z}{2}\frac{z - e^{i\omega} + z - e^{-i\omega}}{(z - e^{i\omega})(z - e^{-i\omega})}$$

$$= \frac{z^2 - z\cos\omega}{z^2 - 2z\cos\omega + 1}$$

The signal $\sin\omega n$

Using equality $2i\sin\alpha = e^{i\alpha} - e^{-i\alpha}$ where i is the imaginary unit for complex numbers, now:

$$\mathcal{Z}\left\{\sin\omega n\right\}(z) = \sum_{n=0}^{\infty}\sin\omega n\, z^{-n} = \frac{1}{2i}\sum_{n=0}^{\infty}(e^{i\omega n} - e^{-i\omega n})z^{-n}$$

$$= \frac{1}{2i}\sum_{n=0}^{\infty}(e^{i\omega}z^{-1})^n - \frac{1}{2i}\sum_{n=0}^{\infty}(e^{-i\omega}z^{-1})^n$$

$$= \frac{1}{2i}\frac{1}{1-e^{i\omega}z^{-1}} - \frac{1}{2i}\frac{1}{1-e^{-i\omega}z^{-1}}$$

$$= \frac{z}{2i}\frac{z-e^{-i\omega}-z+e^{i\omega}}{(z-e^{i\omega})(z-e^{-i\omega})}$$

$$= \frac{z\sin\omega}{z^2-2z\cos\omega+1}$$

The signal n^k (1/2)

Using equality

$$-z\frac{\mathrm{d}}{\mathrm{d}z}z^{-n} = nz^{-n}$$

applying k times

$$\left(-z\frac{\mathrm{d}}{\mathrm{d}z}\right)^k z^{-n} = \underbrace{\left(-z\frac{\mathrm{d}}{\mathrm{d}z}\right)\cdots\left(-z\frac{\mathrm{d}}{\mathrm{d}z}\right)}_{k \text{ volte}} z^{-n} = n^k z^{-n}$$

using this equality in the transform $\mathcal{Z}\left\{f_n n^k\right\}(z)$:

$$\mathcal{Z}\left\{f_n n^k\right\}(z) = \sum_{n=0}^{\infty} f_n n^k z^{-n} = \sum_{n=0}^{\infty} f_n \left(-z\frac{\mathrm{d}}{\mathrm{d}z}\right)^k z^{-n}$$

$$= \left(-z\frac{\mathrm{d}}{\mathrm{d}z}\right)^k \sum_{n=0}^{\infty} f_n z^{-n} = (-1)^k \left(z\frac{\mathrm{d}}{\mathrm{d}z}\right)^k \mathcal{Z}\left\{f_n\right\}(z)$$

The signal n^k (2/2)

Using the rule

$$\mathcal{Z}\left\{f_n n^k\right\}(z) = (-1)^k \left(z\frac{\mathrm{d}}{\mathrm{d}z}\right)^k \mathcal{Z}\{f_n\}(z)$$

and apply it with $f_n = \mathbf{1}_n$:

$$\begin{aligned} \mathcal{Z}\left\{n^k\right\}(z) &= (-1)^k \left(z\frac{\mathrm{d}}{\mathrm{d}z}\right)^k \mathcal{Z}\{\mathbf{1}_n\}(z) \\ &= (-1)^k \left(z\frac{\mathrm{d}}{\mathrm{d}z}\right)^k \frac{z}{z-1} \end{aligned}$$

Z-transform of an important signal (1/2)

Consider the signal $n_k = n(n-1)\cdots(n-k+1)$ and observe that

$$\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^k z^{-(n-k+1)} = (-1)^k n_k z^{-n-1}$$

using this last equality

$$\begin{aligned}\mathcal{Z}\left\{f_{n-k} n_k\right\}(z) &= \sum_{n=0}^{\infty} f_{n-k} n_k z^{-n} \\ &= \sum_{n=0}^{\infty} f_{n-k} (-1)^k z \left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^k z^{-(n-k+1)} \\ &= (-1)^k z \left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^k \left[\frac{1}{z} \sum_{n=0}^{\infty} f_{n-k} z^{-(n-k)}\right]\end{aligned}$$

Z-transform of an important signal (2/2)

Observe that $f_n = 0$ for $n < 0$, thus the signal $n_k = n(n-1)\cdots(n-k+1)$ satisfy:

$$\mathcal{Z}\left\{f_{n-k}n_k\right\}(z) = (-1)^k z\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^k\left[\frac{1}{z}\sum_{m=-k}^{\infty} f_m z^{-m}\right]$$

$$= (-1)^k z\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^k\left[\frac{1}{z}\mathcal{Z}\left\{f_n\right\}(z)\right]$$

Tabella delle Trasformate (1/2)	
δ_n	1
$\mathbf{1}_n$	$\dfrac{z}{z-1}$
a^n	$\dfrac{z}{z-a}$
$a^n\, f_n$	$\widetilde{f}\left(\dfrac{z}{a}\right)$
$a^n \dbinom{n}{k}$	$\dfrac{a^k\, z}{(z-a)^{k+1}}$
$n^k f_n$	$(-1)^k\left(z\dfrac{\mathrm{d}}{\mathrm{d}z}\right)^k\widetilde{f}(z)$

Tabella delle Trasformate (2/2)	
f_{n+k}	$z^k\left(\widetilde{f}(z) - \sum_{j=0}^{k-1} f_j z^{-j}\right)$
f_{n-k}	$z^{-k}\widetilde{f}(z)$
$(f \star g)_n$	$\widetilde{f}(z)\widetilde{g}(z)$
$a^n \sin \omega n$	$\dfrac{za\sin\omega}{z^2 - 2za\cos\omega + a^2}$
$a^n \cos \omega n$	$\dfrac{z^2 - za\cos\omega}{z^2 - 2za\cos\omega + a^2}$
$f_{n-k}n_k$	$(-1)^k z \dfrac{\mathrm{d}^k}{\mathrm{d}z^k}\left(\dfrac{1}{z}\widetilde{f}(z)\right)$

Initial and final value theorems

Theorem (Theorem of final value)

If a signal f_n reach a constant limit value, i.e.

$$\lim_{n\to\infty} f_n = f_\infty$$

then

$$f_\infty = \lim_{z\to 1} \frac{z}{z-1}\widetilde{f_n}(z)$$

Theorem (Theorem of initial value)

$$f_0 = \lim_{z\to\infty} \widetilde{f_n}(z)$$

Z-transform standard form

- In many applications Z-transform can be written in the form:

$$\frac{G(z)}{z} = \frac{P(z)}{Q(z)} = \frac{b_0 + b_1 z + b_2 z^2 + \cdots + b_m z^m}{(z - p_1)^{m_1}(z - p_2)^{m_2} \cdots (z - p_n)^{m_n}}$$

  where $p_i \neq p_j$ if $i \neq j$.
- $\partial P(z) < \partial Q(z)$ can always be assumed
- As for the Laplace transform partial fraction expansion can be done:

$$\frac{G(z)}{z} = \sum_{j=1}^{n} \sum_{k=1}^{m_j} \frac{\alpha_{jk}}{(z - p_j)^k}$$

Explicit formula for the solution

When $G(z)$ is written in partial fraction expansion as follows:

$$G(z) = \sum_{j=1}^{n} \sum_{i=1}^{m_j} \alpha_{ji} \frac{z}{(z-p_j)^i}$$

formally inversion is done consulting Z-transform table:

$$G_n = \sum_{j=1}^{n} \sum_{i=1}^{m_j} \alpha_{ji} n_{i-1} p_j^n$$

As for Laplace transform the formula is valid if p_j is a real root not if it is complex. Anyhow when complex root are found they are collected with the corresponding conjugate to find the corresponding signal in the table.

Example: Fibonacci sequence (1/3)

- The sequence is defined recursively as

$$F_{n+2} = F_{n+1} + F_n, \qquad F_0 = F_1 = 1$$

Attention, to avoid initial condition loss use forward shift!

- Using Z-transform with shift rule

$$z^2\widetilde{F}(z) - F_0 z^2 - F_1 z = z\widetilde{F}(z) - F_0 z + \widetilde{F}(z)$$

- Solving transformed equation

$$\widetilde{F}(z) = \frac{F_0 z^2 + (F_1 - F_0)z}{z^2 - z - 1}$$

- using initial conditions: $F_0 = F_1 = 1$

$$\widetilde{F}(z) = \frac{z^2}{z^2 - z - 1}$$

Example: Fibonacci sequence (2/3)

- from factorization

$$z^2 - z - 1 = (z - z_1)(z - z^2), \qquad z_1 = \frac{1+\sqrt{5}}{2}, \; z_2 = \frac{1-\sqrt{5}}{2}$$

- Using partial fraction expansion

$$\frac{\widetilde{F}(z)}{z} = \frac{z}{z^2 - z - 1} = \frac{A}{z - z_1} + \frac{B}{z - z_2}$$

where $A = (5+\sqrt{5})/10$ and $B = (5-\sqrt{5})/10$, now

$$\widetilde{F}(z) = \frac{Az}{z - z_1} + \frac{Bz}{z - z_2}$$

Example: Fibonacci sequence (3/3)

- Using transform rule $\mathcal{Z}\{a^n\}(z) = z/(z-a)$

$$F_n = Az_1^n + Bz_2^n$$

- substituting

$$z_1 = \frac{1+\sqrt{5}}{2}, \quad z_2 = \frac{1-\sqrt{5}}{2}, \quad A = \frac{5+\sqrt{5}}{10}, \quad B = \frac{5-\sqrt{5}}{10}$$

it follows

$$F_n = \frac{5+\sqrt{5}}{10}\left(\frac{1+\sqrt{5}}{2}\right)^n + \frac{5-\sqrt{5}}{10}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

$$= \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}$$

Why use forward shift?

- Consider Fibonacci sequence

$$F_n = F_{n-1} + F_{n-2}, \qquad F_0 = F_1 = 1$$

- For every solution F_n the shift $G_n = F_{n-1}$ and $H_n = F_{n-2}$ are such that $G_0 = 0$ and $H_0 = 0$.
- This imply $F_0 = G_0 + H_0 = 0$ i.e. initial conditions are forced to be 0.
- Practically forward shift can be used only if initial conditions are all 0.
- Alternatively you can use bilateral Z-transfrom

$$\mathcal{Z}\left\{f_n\right\}(z) = \sum_{n=-\infty}^{\infty} f_n z^{-n} \lim_{N\to\infty} \sum_{n=-N}^{N} f_n z^{-n}$$

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