Laplace Transform Inversion

(Computational Methods for Mechatronics [140466])

Enrico Bertolazzi

DII - Dipartimento di Ingegneria Industriale – Università di Trento

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Outline

Bromwich-Mellin or Riemann-Fourier Formula

Theorem (inversion of Laplace Transform)

Let $f(t)$ a function with $\widehat{f}(s)$ and λ_0 as convergence abscissa. Given α a real number such that $\alpha > \lambda_0$ then for the point t where $f(t)$ is continuous:

$$f(t) = \frac{1}{2\pi\imath} \lim_{\beta \to +\infty} \int_{\alpha - j\beta}^{\alpha + \imath\beta} e^{st} \widehat{f}(s)\,\mathrm{d}s$$

for the point t where $f(t)$ has a jump:

$$\frac{f(t+0) + f(t-0)}{2} = \frac{1}{2\pi\imath} \lim_{\beta \to +\infty} \int_{\alpha - j\beta}^{\alpha + \imath\beta} e^{st} \widehat{f}(s)\,\mathrm{d}s$$

The line $x = \alpha$ on the complex plane is denoted Bromwich line. Notice that values cannot depend on α provided that $\alpha > \lambda_0$.

Bromwich-Mellin formula is not practical for Laplace Transform inversion. For example consider inversion of $1/s$ that we known if the Laplace Transform of Heaviside function:

$$\begin{aligned} h(t) &= \frac{1}{2\pi\imath}\lim_{\beta\to+\infty}\int_{\alpha-\imath\beta}^{\alpha+\imath\beta}\frac{e^{st}}{s}\,\mathrm{d}s \\ &= \frac{1}{2\pi\imath}\lim_{\beta\to+\infty}\int_{-\beta}^{\beta}\frac{e^{(\alpha+\imath\gamma)t}}{\alpha+\imath\gamma}\imath d\gamma \\ &= \frac{e^{\alpha t}}{2\pi}\lim_{\beta\to+\infty}\int_{-\beta}^{\beta}\frac{\cos(\gamma t)(\alpha-\imath\gamma)+\sin(\gamma t)(\imath\alpha+\gamma)}{\alpha^2+\gamma^2}d\gamma \end{aligned}$$

using the property that $\sin(\gamma t)$ and $\gamma\cos(\gamma t)$ are odd function respect to γ:

$$h(t) = \frac{e^{\alpha t}}{2\pi}\lim_{\beta\to+\infty}\int_{-\beta}^{\beta}\frac{\alpha\cos(\gamma t)+\gamma\sin(\gamma t)}{\alpha^2+\gamma^2}d\gamma$$

The integrals

$$h(t) = \frac{e^{\alpha t}}{2\pi} \lim_{\beta \to +\infty} \int_{-\beta}^{\beta} \frac{\alpha \cos(\gamma t) + \gamma \sin(\gamma t)}{\alpha^2 + \gamma^2} d\gamma$$

for $t \neq 0$ and

$$h(0) = \frac{1}{2\pi} \lim_{\beta \to +\infty} \int_{-\beta}^{\beta} \frac{\alpha}{\alpha^2 + \gamma^2} d\gamma$$

for $t = 0$ are difficult to compute. Using MAPLE, for example, the following solution is found:

$$h(t) = \begin{cases} 0 & \text{per } t < 0 \\ 1/2 & \text{per } t = 0 \\ 1 & \text{per } t > 0 \end{cases}$$

- The right way to compute previous integrals is by using complex analysis with residuals.
- Also using complex analysis computation is cumbersome.
- In general Transform is easier than Inversion.

For these reasons the inversion based if partial fraction expansion and Laplace Tables if generally the fastest way to do Laplace Transform Inversion.

Standard form of Laplace Transform (1/2)

- In electrical or mechanical applications Laplace Transform takes the form:

$$G(s) = \frac{P(s)}{Q(s)} = \frac{b_0 + b_1 s + b_2 s^2 + \cdots + b_m s^m}{(s-p_1)^{m_1}(s-p_2)^{m_2}\cdots(s-p_n)^{m_n}}$$

  where $p_i \neq p_j$ and $i \neq j$.

- We assume $\partial P(s) < \partial Q(s)$ otherwise using polynomial division with remainder:

$$P(s) = Q(s)A(s) + B(s) \qquad \partial B(s) < \partial Q(s)$$

  and thus

$$\frac{P(s)}{Q(s)} = A(s) + \frac{B(s)}{Q(s)}$$

Standard form of Laplace Transform (2/2)

- Inversion of Laplace Transform of a polynomial

$$A(s) = a_0 + a_1 s + \cdots + a_n s^n$$

formally

$$\mathcal{L}\left\{A(s)\right\}^{-1}(t) = a_0\delta(t) + a_1\delta^{(1)}(t) + \cdots + a_n\delta^{(n)}(t)$$

- The "functions" $\delta^{(k)}(t)$ are the k-th distributional derivative of delta Dirac function with the property:

$$\int_{-\infty}^{\infty} f(t)\delta^{(k)}(t)\,\mathrm{d}t = (-1)^k f^{(k)}(0)$$

- Except unitary impulse ($\delta(t)$) normally impulse derivative is not considered.
- Thus we can reduce the inversion of Laplace Transform when $f(s)$ is a rational function $P(s)/Q(s)$ with $\partial P(s) < \partial Q(s)$.

Simple roots case

Given the rational complex function

$$G(s) = \frac{b_0 + b_1 s + b_2 s^2 + \cdots + b_m s^m}{(s - p_1)(s - p_2)\cdots(s - p_n)} \qquad (m < n)$$

where $p_i \neq p_j$ if $i \neq j$. $G(s)$ can be written as the sum of simple fractions

$$G(s) = \frac{\alpha_1}{s - p_1} + \frac{\alpha_2}{s - p_2} + \cdots + \frac{\alpha_n}{s - p_n}$$

where:

$$\alpha_i = \lim_{s \to p_i} (s - p_i) G(s)$$

in fact

$$(s - p_i) G(s) = \alpha_i + \sum_{j \neq i} \alpha_j \frac{s - p_i}{s - p_j}$$

Multiple root case

In case of multiple roots, in particular when there is a sinple root p with moltepicity n

$$G(s) = \frac{b_0 + b_1 s + b_2 s^2 + \cdots + b_m s^m}{(s-p)^n} \qquad (m < n)$$

$G(s)$ can be rewritten as the sum of simple fractions as follows

$$G(s) = \frac{\alpha_1}{s-p} + \frac{\alpha_2}{(s-p)^2} + \cdots + \frac{\alpha_n}{(s-p)^n}$$

where $(0! = 1)$:

$$\alpha_{n-k} = \frac{1}{k!} \lim_{s \to p} \frac{\mathrm{d}^k}{\mathrm{d}s^k} [(s-p)^n G(s)], \qquad k = 0, 1, \ldots, n-1$$

in fact

$$(s-p_i)^n G(s) = \alpha_1 (s-p)^{n-1} + \cdots + \alpha_{n-1}(s-p) + \alpha_n$$

General case

In the general case:

$$G(s) = \frac{b_0 + b_1 s + b_2 s^2 + \cdots + b_m s^m}{(s-p_1)^{m_1}(s-p_2)^{m_2}\cdots(s-p_k)^{m_n}} \quad (m < m_1 + m_2 + \cdots + m_n)$$

where m_i is the multiplicity of root p_i. $G(s)$ is then rewritten as the sum of simple fraction as follows

$$G(s) = \sum_{j=1}^{n} \sum_{k=1}^{m_j} \frac{\alpha_{jk}}{(s-p_j)^k}$$

where $(0! = 1)$:

$$\alpha_{j,m_j-k} = \frac{1}{k!} \lim_{s \to p_j} \frac{\mathrm{d}^k}{\mathrm{d}s^k}\left[(s-p_j)^{m_j} G(s)\right], \qquad k = 0, 1, \ldots, m_j - 1$$

Explicit formula for partial fraction expansion

Let $G(s)$ written as the sum of simple fractions

$$G(s) = \sum_{j=1}^{n} \sum_{k=1}^{m_j} \frac{\alpha_{jk}}{(s - p_j)^k}$$

formally the inverse of Laplace Transform of $G(s)$ by looking of the Laplace Transform tables is

$$G(t) = \mathcal{L}\left\{G(s)\right\}^{-1}(t) = \sum_{j=1}^{n} \sum_{k=1}^{m_j} \frac{\alpha_{jk}}{(k-1)!} e^{p_j t} t^{k-1}$$

Attention, in this expression p_j can be a complex number and thus the corresponding function is a complex function, but, Laplace Transform is defined for real value functiuons.

Lemma

Let $p_j = \overline{p}_i$ *then* $m_i = m_j$, *moreover*

$$\alpha_{jk} = \overline{\alpha}_{ik} \qquad k = 1, 2, \ldots, m_i$$

First of all observe that $\alpha_{jm_j} = \overline{\alpha_{im_i}}$:

$$\begin{aligned}\alpha_{jm_j} &= \lim_{s \to p_j} (s - p_j)^{m_j} G(s) = \overline{\lim_{s \to p_j} (\overline{s} - \overline{p}_j)^{m_j} G(\overline{s})} \\ &= \overline{\lim_{s \to \overline{p}_i} (\overline{s} - p_i)^{m_i} G(\overline{s})} = \overline{\lim_{s \to p_i} (s - p_i)^{m_i} G(s)} = \overline{\alpha_{im_j}}\end{aligned}$$

and analogously for the other coefficients

$$\begin{aligned}\alpha_{jm_j - k} &= \lim_{s \to p_j} \frac{\mathrm{d}^k[(s - p_j)^{m_j} G(s)]}{\mathrm{d}s^k} = \lim_{s \to p_j} \overline{\frac{\mathrm{d}^k[(\overline{s} - \overline{p}_j)^{m_j} G(\overline{s})]}{\mathrm{d}s^k}} \\ &= \lim_{s \to \overline{p}_i} \overline{\frac{\mathrm{d}^k[(\overline{s} - p_i)^{m_i} G(\overline{s})]}{\mathrm{d}s^k}} = \lim_{s \to p_i} \overline{\frac{\mathrm{d}^k[(s - p_i)^{m_i} G(s)]}{\mathrm{d}s^k}} = \overline{\alpha_{im_j - k}}\end{aligned}$$

From the previous lemma if the pole of $G(s)$ are real the corresponding coefficients in partial fraction expansion are real. Let $G(s)$ with n roots where complex conjugate roots are counted one time

$$G(s) = \sum_{j=1}^{n} \sum_{k=1}^{m_j} \frac{1}{2} \left[\frac{\beta_{jk}}{(s - p_j)^k} + \frac{\overline{\beta_{jk}}}{(s - \overline{p_j})^k} \right]$$

where

$$\beta_{jk} = \begin{cases} \alpha_{jk} & \text{if } \alpha_{jk} \text{ is a real number} \\ 2\alpha_{jk} & \text{if } \alpha_{jk} \text{ is a complex number} \end{cases}$$

and thus, Laplace inversion becomes

$$G(t) = \mathcal{L}\left\{G(s)\right\}^{-1}(t) = \sum_{j=1}^{n} \sum_{k=1}^{m_j} \frac{\beta_{jk} e^{p_j t} + \overline{\beta_{jk}} e^{\overline{p_j} t}}{2(k-1)!} t^{k-1}$$

consider now $f(t) = \beta e^{pt} + \overline{\beta} e^{\overline{p}t}$ where

$$\beta = a + \imath b, \qquad p = \gamma + \imath\omega,$$

then

$$\begin{aligned} f(t) &= (a+\imath b)e^{(\gamma+\imath\omega)t} + (a-\imath b)e^{(\gamma+\imath\omega)t} \\ &= e^{\gamma t}\Big[(a+\imath b)(\cos(\omega t) + \imath\sin(\omega t)) \\ &\qquad +(a-\imath b)(\cos(\omega t) - \imath\sin(\omega t))\Big] \\ &= e^{\gamma t}\Big[2a\cos(\omega t) - 2b\sin(\omega t)\Big] \end{aligned}$$

and in general

$$f(t) = 2e^{\,\mathrm{Re}(p)t}\Big[\,\mathrm{Re}\,(\beta)\cos(\,\mathrm{Im}\,(p)\,t) - \,\mathrm{Im}\,(\beta)\sin(\,\mathrm{Im}\,(p)\,t)\Big]$$

Practical computation of coefficients α_{ki} (1/2)

- In general if $G(s) = P(s)/Q(s)$ with $\partial P(s) \leq \partial Q(s)$ and $Q(s)$ with n distinct roots p_k with multiplicity m_k (conjugate complex root are counted as a single root) will be written as:

$$G(s) = \sum_{j=1}^{n} \sum_{k=1}^{m_j} \frac{1}{2} \left[\frac{\beta_{jk}}{(s - p_j)^k} + \frac{\overline{\beta}_{jk}}{(s - \overline{p}_j)^k} \right]$$

- Using previous results by set $r_k = \mathrm{Re}\,(p_k)$ and $\omega_k = \mathrm{Im}\,(p_k)$ it follows

$$G(t) = \sum_{j=1}^{n} e^{r_j t} \left[\mathrm{Re}\,(P_j(t)) \cos(\omega_j t) - \mathrm{Im}\,(P_j(t)) \sin(\omega_j t) \right]$$

where

$$P_j(t) = \sum_{k=1}^{m_j} t^{k-1} \frac{\beta_{jk}}{(k-1)!}$$

Practical computation of coefficients α_{ki} (2/2)

Multiplying previous expansion by $Q(s)$ the following polynomial equality is obtained

$$P(s) = Q(s)G(s) = \sum_{k=1}^{n}\sum_{i=1}^{m_k} \frac{Q(s)}{2}\left[\frac{\alpha_{ki}}{(s-p_k)^i} + \frac{\overline{\alpha}_{ki}}{(s-\overline{p}_k)^i}\right]$$

Coefficients α_{km_k} are obtained from the relation

$$P(p_k) = Q(p_k)G(p_k)$$

the other coefficients are obtained by differentiation

$$\alpha_{j,m_k-j} := \text{risolve per } \alpha_{j,m_k-j} \qquad \frac{\mathrm{d}^j}{\mathrm{d}s^j}P(s)\Big|_{s=p_k} = 0$$

A complex example (1/10)

Find partial fraction expansion of the following rational polynomial

$$\frac{s^2+s+1}{(s-1)(s-3)^3(s-(1+i))^2(s-(1-i))^2}$$

the roots are $p_1 = 1$, $p_2 = 3$, $p_3 = 1+i$ with multiplicity $m_1 = 1$, $m_2 = 3$ and $m_3 = 2$. Partial fraction expansion takes the form (factors $1/2$ are putted on the tail)

$$\begin{aligned} G(s) &= \frac{a}{s-1} + \frac{b_1}{s-3} + \frac{b_2}{(s-3)^2} + \frac{b_3}{(s-3)^3} \\ &+ \frac{c_1+id_1}{s-(1+i)} + \frac{c_1-id_1}{s-(1-i)} \\ &+ \frac{c_2+id_2}{(s-(1+i))^2} + \frac{c_2-id_2}{(s-(1-i))^2} \end{aligned}$$

A complex example (2/10)

Compute

$$
\begin{aligned}
P(s)G(s) &= a(s-3)^3(s-(1+i))^2(s-(1-i))^2 \\
&+ b_1(s-1)(s-3)^2(s-(1+i))^2(s-(1-i))^2 \\
&+ b_2(s-1)(s-3)(s-(1+i))^2(s-(1-i))^2 \\
&+ b_3(s-1)(s-(1+i))^2(s-(1-i))^2 \\
&+ (c_1+id_1)(s-1)(s-3)^3(s-(1+i))(s-(1-i))^2 \\
&+ (c_1-id_1)(s-1)(s-3)^3(s-(1+i))^2(s-(1-i)) \\
&+ (c_2+id_2)(s-1)(s-3)^3(s-(1-i))^2 \\
&+ (c_2-id_2)(s-1)(s-3)^3(s-(1+i))^2
\end{aligned}
$$

A complex example (3/10)

Polynomial expansion

$$
\begin{aligned}
Q(s)G(s) = & \left(a + b_1 + 2\,c_1\right)s^7 \\
& + (2\,c_2 - 11\,b_1 + b_2 - 13\,a - 26\,c_1 - 2\,d_1)\,s^6 \\
& + (-24\,c_2 - 8\,b_2 + b_3 - 4\,d_2 + 51\,b_1 + 71\,a + 24\,d_1 + 140\,c_1)\,s^5 \\
& + (-116\,d_1 - 215\,a - 133\,b_1 - 408\,c_1 + 27\,b_2 - 5\,b_3 + 44\,d_2 + 112\,c_2)\,s^4 \\
& + (12\,b_3 + 706\,c_1 + 400\,a - 52\,b_2 + 292\,d_1 + 216\,b_1 - 184\,d_2 - 252\,c_2)\,s^3 \\
& + (60\,b_2 - 738\,c_1 - 16\,b_3 - 468\,a + 270\,c_2 - 220\,b_1 - 414\,d_1 + 360\,d_2)\,s^2 \\
& + (324\,a + 12\,b_3 + 132\,b_1 - 108\,c_2 - 40\,b_2 + 432\,c_1 - 324\,d_2 + 324\,d_1)\,s \\
& -108\,a - 36\,b_1 + 12\,b_2 - 4\,b_3 - 108\,c_1 - 108\,d_1 + 108\,d_2
\end{aligned}
$$

notice that the polynomial as real coefficients in s.

A complex example (4/10)

Evaluating at $s = 1$ (the first root)

$$Q(1)G(1) = -8a, \qquad P(1) = 3, \qquad a = -\frac{3}{8}$$

Evaluating at $s = 3$ (the second root)

$$Q(3)G(3) = 50b_3, \qquad P(3) = 13, \qquad b_3 = \frac{13}{50}$$

Evaluating at $s = 1 + i$ (the third root)

$$Q(1+i)G(1+i) = 44c_2 - 8d_2 + i(8c_2 + 44d_2), \quad P(1+i) = 2 + 3i,$$

then the following linear system is obtained

$$44c_2 - 8d_2 = 2, \qquad 8c_2 + 44d_2 = 3.$$

which solution is $c_2 = 7/125$ e $d_2 = 29/500$.

A complex example (5/10)

Compute

$$\frac{\mathrm{d}}{\mathrm{d}s}(Q(s)G(s)) = 7\ (a + b_1 + 2\,c_1)\,s^6$$

$$+6\ (2\,c_2 - 11\,b_1 + b_2 - 13\,a - 26\,c_1 - 2\,d_1)\,s^5$$

$$+5\ (-24\,c_2 - 8\,b_2 + b_3 - 4\,d_2 + 51\,b_1 + 71\,a + 24\,d_1 + 140\,c_1)\,s^4$$

$$+4\ (-116\,d_1 - 215\,a - 133\,b_1 - 408\,c_1 + 27\,b_2 - 5\,b_3 + 44\,d_2 + 112\,c_2)\,s^3$$

$$+3\ (12\,b_3 + 706\,c_1 + 400\,a - 52\,b_2 + 292\,d_1 + 216\,b_1 - 184\,d_2 - 252\,c_2)\,s^2$$

$$+2\ (60\,b_2 - 738\,c_1 - 16\,b_3 - 468\,a + 270\,c_2 - 220\,b_1 - 414\,d_1 + 360\,d_2)\,s$$

$$+324\,a + 12\,b_3 + 132\,b_1 - 108\,c_2 - 40\,b_2 + 432\,c_1 - 324\,d_2 + 324\,d_1$$

A complex example (6/10)

Evaluating at $s = 3$ (second root)

$$\frac{\mathrm{d}}{\mathrm{d}s} Q(s)G(s)|_{s=3} = 50b_2 + 105b_3, \qquad P'(3) = 7,$$

using previously computed value $b_3 = -\frac{13}{50}$ then $b_2 = -203/500$.
Evaluating at $s = 1 + i$ (the third root)

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s} Q(s)G(s)|_{s=1+i} &= 44\,c_1 - 8\,d_1 - 32\,c_2 + 124\,d_2 \\ &+ 8\,ic_1 + 44\,id_1 - 32\,id_2 - 124\,ic_2 \\ \frac{\mathrm{d}}{\mathrm{d}s} P(s)|_{s=1+i} &= 3 + 2i\end{aligned}$$

solving the linear system with the values of c_2 e d_2 the values $c_1 = -6/625$ e $d_1 = 309/1250$ are computed.

A complex example (7/10)

Compute

$$\frac{d^2}{ds^2}(Q(s)G(s)) = (42\,a + 42\,b_1 + 84\,c_1)\,s^5$$

$$+ (60\,c_2 - 330\,b_1 + 30\,b_2 - 390\,a - 780\,c_1 - 60\,d_1)\,s^4$$

$$+ (1420\,a - 480\,c_2 + 1020\,b_1 + 480\,d_1 + 2800\,c_1 + 20\,b_3 - 160\,b_2 - 80\,d_2)\,s^3$$

$$+ (-4896\,c_1 + 1344\,c_2 + 528\,d_2 - 2580\,a - 60\,b_3 - 1596\,b_1 - 1392\,d_1 + 324\,b_2)\,s^2$$

$$+ (72\,b_3 - 1512\,c_2 + 4236\,c_1 + 2400\,a + 1296\,b_1 - 312\,b_2 + 1752\,d_1 - 1104\,d_2)\,s$$

$$-936\,a + 540\,c_2 - 440\,b_1 + 720\,d_2 - 32\,b_3 - 828\,d_1 - 1476\,c_1 + 120\,b_2$$

A complex example (8/10)

Evaluating at $s = 3$ (the second root)

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2} Q(s)G(s)|_{s=3} = 100b_1 + 210b_2 + 184b_3, \qquad P''(3) = 2,$$

using previously computed values $b_3 = -\frac{13}{50}$ and $b_2 = -203/500$ the value $b_1 = 1971/5000$ is obtained.

Esempio Pratico molto complesso (9/10)

Putting all thing together

$$G(s) = \frac{-3/8}{s-1} + \frac{1971}{5000(s-3)} + \frac{-203}{500(s-3)^2} + \frac{13}{50(s-3)^3}$$
$$+ \frac{-6/625 + 309/1250i}{s-(1+i)} + \frac{-6/625 - 309/1250i}{s-(1-i)}$$
$$+ \frac{7/125 + 29/500i}{(s-(1+i))^2} + \frac{7/125 - 29/500i}{(s-(1-i))^2}$$

A complex example (10/10)

And using simple relation on complex number and complex roots

$$G(t) = e^{t}\,\frac{(560\,t - 96)\cos(t) - (2472 + 580\,t)\sin(t) - 1875}{5000}$$
$$+e^{3\,t}\,\frac{1971 - 2030\,t + 650\,t^2}{5000}$$

Riferimenti

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