Laplace Transform

(Computational Methods for Mechatronics [140466])

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Outline

Pierre-Simon Laplace, 1749-1827

Laplace Transform

- Definition

$$f(t) \rightarrow \widehat{f}(s) = \mathcal{L}\left\{f(t)\right\}(s)$$

$$\widehat{f}(s) = \int_{0^-}^{+\infty} f(t)e^{-st}\,\mathrm{d}t = \lim_{\epsilon \to 0^+} \lim_{M \to +\infty} \int_{-\epsilon}^{M} f(t)e^{-st}\,\mathrm{d}t$$

- Usefulness: transform

Differential equations $\Rightarrow$ Algebraic equations

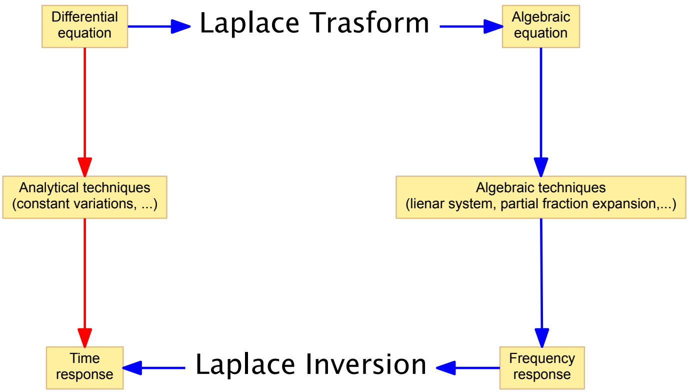
- Logarithm analogy:

$$a \rightarrow \log a$$

$$a \cdot b \rightarrow \log a + \log b$$

i.e. logarithm convert products into additions which are easier to manipulate.

Laplace Transform as a tool for ODE solution



Laplace Transform properties

Table 1			
Linearity	$a\,f(t) + b\,g(t)$	$a\,\widehat{f}(s) + b\,\widehat{g}(s)$	1
Scale change	$f(at)$	$\dfrac{1}{a}\,\widehat{f}\left(\dfrac{s}{a}\right)$	2
Translation respect to s	$e^{at} f(t)$	$\widehat{f}(s-a)$	3
Translation respect to t	$f(t-a)$	$e^{-as}\widehat{f}(s)$	4

a and b are real number. Moreover $a > 0$ for point 2 and 4.

$$\begin{aligned}\mathcal{L}\left\{af(t)+bg(t)\right\}(s) &= \int_{0^-}^{+\infty}(af(t)+bg(t))e^{-st}\,\mathrm{d}t\\ &= a\int_{0^-}^{+\infty}f(t)e^{-st}\,\mathrm{d}t + b\int_{0^-}^{+\infty}g(t)e^{-st}\,\mathrm{d}t\\ &= a\,\widehat{f}(s)+b\,\widehat{g}(s)\end{aligned}$$

$$\begin{aligned}\mathcal{L}\left\{f(at)\right\}(s) &= \int_{0^-}^{+\infty}f(at)e^{-st}\,\mathrm{d}t \qquad [t=z/a, \quad a>0]\\ &= \int_{0^-}^{+\infty}f(z)e^{-sz/a}\frac{\mathrm{d}z}{a}\\ &= \frac{1}{a}\widehat{f}\left(\frac{s}{a}\right)\end{aligned}$$

$$\mathcal{L}\left\{e^{at}f(t)\right\}(s) = \int_{0^-}^{+\infty} e^{at}f(t)e^{-st}\,\mathrm{d}t = \int_{0^-}^{+\infty} f(t)e^{(a-s)t}\,\mathrm{d}t$$
$$= \widehat{f}(s-a)$$

$$\mathcal{L}\left\{f(t-a)\right\}(s) = \int_{0^-}^{+\infty} f(t-a)e^{-st}\,\mathrm{d}z \qquad [t-a=z]$$
$$= \int_{-a}^{+\infty} f(z)e^{-s(z+a)}\,\mathrm{d}z \qquad [f(z)=0 \text{ per } z\leq 0]$$
$$= e^{-sa}\int_{0}^{+\infty} f(z)e^{-sz}\,\mathrm{d}z$$
$$= e^{-as}\widehat{f}(s)$$

When Laplace transform exists? (1/3)

- Not all function have a Laplace Transform, for example

$$\mathcal{L}\left\{e^{t^2}\right\}(s) = \int_{0^-}^{+\infty} e^{t^2-st}\,\mathrm{d}t$$
$$= \int_{0^-}^{T} e^{(t-s)t}\,\mathrm{d}t + \int_{T}^{+\infty} e^{(t-s)t}\,\mathrm{d}t$$

for all possible s choose $T > \mathrm{Re}(s)$ so that

$$\int_{T}^{+\infty} e^{(t-s)t}\,\mathrm{d}t$$

is not convergent. Thus, the function have not a Laplace Transform for any $s \in \mathbb{C}$.

When Laplace transform exists? $(2/3)$

Let be $f(t)$ continuous with bounds: $|f(t)| \leq Me^{Nt}$ for $t \geq T$ then the function have a Laplace Transform:

$$\mathcal{L}\{f\}(s) = \int_{0^-}^{T} f(t)e^{-st}\,\mathrm{d}t + \int_{T}^{+\infty} f(t)e^{-st}\,\mathrm{d}t$$

In fact,

$$\left|\int_{T}^{+\infty} f(t)e^{-st}\,\mathrm{d}t\right| \leq \int_{T}^{+\infty} \left|f(t)e^{-st}\right|\,\mathrm{d}t \leq \int_{T}^{+\infty} Me^{Nt}\left|e^{-st}\right|\,\mathrm{d}t$$
$$= \int_{T}^{+\infty} Me^{Nt}e^{-\operatorname{Re}(s)t}\,\mathrm{d}t = M\int_{T}^{+\infty} e^{(N-\operatorname{Re}(s))t}\,\mathrm{d}t$$

and for $\operatorname{Re}(s) > N$ hold

$$\lim_{T\to+\infty}\int_{T}^{+\infty} e^{(N-\operatorname{Re}(s))t}\,\mathrm{d}t = 0$$

When Laplace transform exists? (3/3)

Definition (Piecewise continuous function)

$f(t)$ is a piecewise continuous function if for all interval $[0, T]$

- *is discontinuous at most on a finite number of points*
- *il finitely bounded*

Definition (Exponential order function)

$f(t)$ is an exponential order function if is piecewise continuous with bound:

$$|f(t)| \leq Me^{Nt} \qquad per \quad t \geq T$$

From now forward we assume the considered functions are of exponential order with piecewise continuous derivative up to the required order.

Theorem (1)

Let $f(t)$ of exponential order, then:

$$\lim_{s\to\infty} \widehat{f}(s) = 0, \qquad s \in \mathbb{R}$$

Proof: Assuming s real

$$\left|\widehat{f}(s)\right| = \left|\int_{0^-}^{\infty} f(t)e^{-st}\,\mathrm{d}t\right| \le \int_{0^-}^{\infty} |f(t)|\, e^{-st}\,\mathrm{d}t$$
$$\le M \int_{0^-}^{\infty} e^{(N-s)t}\,\mathrm{d}t = \frac{M}{s-N}$$

but

$$\lim_{s\to+\infty} \frac{M}{s-N} = 0$$

Polynomial and exponential growth

- Heaviside function

$$u(t) = \begin{cases} 0 & \text{se } t < 0; \\ 1 & \text{se } t \geq 0. \end{cases}$$

- Linear growth

$$t_+ = t\, u(t) = \begin{cases} 0 & \text{se } t < 0; \\ t & \text{se } t \geq 0. \end{cases}$$

- Polynomial growth

$$t_+^k = t^k\, u(t) = \begin{cases} 0 & \text{se } t < 0; \\ t^k & \text{se } t \geq 0. \end{cases}$$

- Esponenziale growth

$$v(t) = a^{bt}\, u(t) = \begin{cases} 0 & \text{se } t < 0; \\ a^{bt} & \text{se } t \geq 0. \end{cases}$$

Table 2		
1	$\frac{1}{s}$	5
t	$\frac{1}{s^2}$	6
t^k	$\frac{k!}{s^{k+1}}$	7
a^{bt}	$\frac{1}{s - b\log a}$	8

Attention, functions on the first column shall be deemed equal to 0 for $t < 0$, i.e. $f(t) \rightarrow \widehat{f}(s)$ or $u(t)f(t) \rightarrow \widehat{f}(s)$ where $u(t)$ is the Heaviside function.

- Heaviside function

$$u(t) = \begin{cases} 0 & \text{se } t < 0; \\ 1 & \text{se } t \geq 0. \end{cases}$$

- Laplace Transform (assuming $\mathrm{Re}(s) > 0$):

$$\mathcal{L}\{u\}(s) = \widehat{u}(s) = \int_{0^-}^{+\infty} u(t) e^{-st} \, \mathrm{d}t = \int_{0^-}^{+\infty} e^{-st} \, \mathrm{d}t$$

$$= \left[-\frac{1}{s} e^{-st} \right]_{0^-}^{+\infty} = \frac{1}{s}$$

- Linear growth

$$t_+ = t\,u(t)$$

- Laplace Transform (assuming $\mathrm{Re}\,(s) > 0$):

$$\begin{aligned}
\mathcal{L}\left\{t_+\right\}(s) = \widehat{t_+}(s) &= \int_{0^-}^{+\infty} t\,u(t)e^{-st}\,\mathrm{d}t = \int_{0^-}^{+\infty} te^{-st}\,\mathrm{d}t \\
&= \left[-\frac{t}{s}e^{-st}\right]_{0^-}^{+\infty} + \frac{1}{s}\int_{0^-}^{+\infty} e^{-st}\,\mathrm{d}t \\
&= 0 + \frac{1}{s}\left[-\frac{1}{s}e^{-st}\right]_{0^-}^{+\infty} \\
&= \frac{1}{s^2}
\end{aligned}$$

- Polynomial growth

$$t_+^k = t^k\, u(t)$$

- Laplace Transform (assuming $\mathrm{Re}\,(s) > 0$):

$$\begin{aligned}
\mathcal{L}\left\{t_+^k\right\}(s) = \widehat{t_+^k}(s) &= \int_{0^-}^{+\infty} t^k\, u(t) e^{-st}\,\mathrm{d}t = \int_{0^-}^{+\infty} t^k e^{-st}\,\mathrm{d}t \\
&= \left[-\frac{t^k}{s} e^{-st}\right]_{0^-}^{+\infty} + \frac{k}{s}\int_{0^-}^{+\infty} t^{k-1} e^{-st}\,\mathrm{d}t \\
&= 0 + \frac{k}{s}\,\widehat{t_+^{k-1}}(s)
\end{aligned}$$

- Using induction and noticing that $\widehat{t_+}(s) = \dfrac{1}{s^2}$ it follows

$$\widehat{t_+^k}(s) = \frac{k!}{s^{k+1}}$$

- Exponential growth

$$v(t) = a^{bt}\, u(t)$$

- Laplace Transform (assuming $\operatorname{Re}(s) > b \log a$):

$$\begin{aligned}
\mathcal{L}\left\{a^{bt}\right\}(s) &= \int_{0^-}^{+\infty} a^{bt}\, u(t) e^{-st}\, \mathrm{d}t = \int_{0^-}^{+\infty} a^{bt} e^{-st}\, \mathrm{d}t \\
&= \int_{0^-}^{+\infty} e^{bt \log a} e^{-st}\, \mathrm{d}t = \int_{0^-}^{+\infty} e^{(b \log a - s)t}\, \mathrm{d}t \\
&= \left[\frac{1}{(b \log a - s)} e^{(b \log a - s)t}\right]_{0^-}^{+\infty} \\
&= \frac{1}{s - b \log a}
\end{aligned}$$

First derivative Laplace Transform (1/2)

Theorem (First derivative Laplace Transform)

Let $f(t)$ of exponential order with piecewise continuous first derivative. The Laplace Transform of $f'(t)$ becomes:

$$\mathcal{L}\left\{f'(t)\right\}(s) = s\widehat{f}(s) - f(0^+)$$

(assuming $f(t) = 0$ for $t \leq 0$)

Proof: Let $\operatorname{Re}(s) > 0$ and $\beta > 0$:

$$\int_\beta^{+\infty} f'(t)e^{-st}\,\mathrm{d}t = \left[f(t)e^{-st}\right]_\beta^{+\infty} + s\int_\beta^{+\infty} f(t)e^{-st}\,\mathrm{d}t$$

$$= -f(\beta)e^{-s\beta} + s\int_\beta^{+\infty} f(t)e^{-st}\,\mathrm{d}t$$

First derivative Laplace Transform

(2/2)

and thus,

$$\begin{aligned}\int_{-\epsilon}^{+\infty} f'(t)e^{-st}\,\mathrm{d}t &= \lim_{\beta\to 0}\left[\int_{-\epsilon}^{0} f'(t)e^{-st}\,\mathrm{d}t + \int_{\beta}^{+\infty} f'(t)e^{-st}\,\mathrm{d}t\right] \\ &= \lim_{\beta\to 0}\left[-f(\beta)e^{-s\beta} + s\int_{\beta}^{+\infty} f(t)e^{-st}\,\mathrm{d}t + 0\right] \\ &= -f(0^+) + s\int_{0^+}^{+\infty} f(t)e^{-st}\,\mathrm{d}t\end{aligned}$$

from $f(t) = 0$ for $t \leq 0$ it follows $\int_{-\epsilon}^{0} f(t)e^{-st}\,\mathrm{d}t = 0$ and

$$\mathcal{L}\left\{f'(t)\right\}(s) = -f(0^+) + s\int_{0^-}^{+\infty} f(t)e^{-st}\,\mathrm{d}t.$$

k-th derivative Laplace Transform

Theorem (k-th derivative Laplace Transform)

Let $f(t)$ of exponential order up to $k-1$-derivative and k-th derivative piecewise continuous. Then Laplace Transform of k-th derivative become:

$$\mathcal{L}\left\{f^{(k)}(t)\right\}(s) = s^k \widehat{f}(s) - \sum_{i=0}^{k-1} s^i f^{(k-i-1)}(0^+).$$

(assuming $f(t) = 0$ for $t \leq 0$)

Proof: Is similar to the proof for first derivative using k-times integration by part.

Laplace Transform of an integral

Theorem (Laplace Transform of an integral)

Let $f(t)$ *piecewise continuous and* $g(t)$ *defined as*

$$g(t) = \int_0^t f(z)\,\mathrm{d}z$$

Laplace transform $\mathcal{L}\left\{g(t)\right\}(s) = \widehat{g}(s)$ *become:*

$$\widehat{g}(s) = \frac{1}{s}\widehat{f}(s).$$

Proof: Apply derivation rule for the function $g(t)$ and observe that $g'(t) = f(t)$ and $g(0) = 0$.

Initial and final value

Theorem (of the initial value)

Let $f(t)$ of exponential order with piecewise continuous first derivative, then:

$$f(0^+) = \lim_{s \to +\infty} s\widehat{f}(s) \qquad s \in \mathbb{R}$$

Proof: From theorem 1 with $f'(t)$

$$0 = \lim_{s \to +\infty} \mathcal{L}\left\{f'(t)\right\}(s) = \lim_{s \to +\infty} s\widehat{f}(s) - f(0^+)$$

Theorem (of the final value)

Let $f(t)$ of exponential order with piecewise continuous first derivative, if the limit $f(+\infty) = \lim_{t\to+\infty} f(t)$ exists then:

$$f(+\infty) = \lim_{s\to 0} s\widehat{f}(s) \qquad s \in \mathbb{R}$$

Proof: Using Laplace Transform of $f'(t)$

$$\lim_{s\to 0^+} \mathcal{L}\left\{f'(t)\right\}(s) = \lim_{s\to 0^+} s\widehat{f}(s) - f(0^+)$$

$$\lim_{s\to 0^+} \mathcal{L}\left\{f'(t)\right\}(s) = \lim_{s\to 0^+} \int_{0^-}^{\infty} f'(t)e^{-st}\,\mathrm{d}t = \int_{0^-}^{\infty} f'(t) \lim_{s\to 0^+} e^{-st}\,\mathrm{d}t$$

$$= \int_{0^-}^{\infty} f'(t)\,\mathrm{d}t = f(+\infty) - f(0^+)$$

Here we use Lebesgue's dominated convergence theorem.

- Moltiply by t^n

$$\mathcal{L}\left\{t^n f(t)\right\}(s) = (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}s^n}\widehat{f}(s)$$

- Division by t. Let $g(t) = tf(t)$ then from the previous formula

$$\mathcal{L}\left\{g(t)\right\}(s) = -\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{L}\left\{f(t)\right\}(s)$$

that can be written as: $\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{L}\left\{\frac{g(t)}{t}\right\}(s) = -\widehat{g}(s)$ or better

$$\mathcal{L}\left\{\frac{g(t)}{t}\right\}(s) = -\int \widehat{g}(s)\,\mathrm{d}s + C = \widehat{h}(s)$$

Complex constant C must be chosen such that $\widehat{h}(s)$ satisfy initial and final value theorem. Obviously $\lim_{t\to 0^+} g(t)/t$ must exists and must be finite.

Theorem (Periodic function Laplace Transform)

Let $f(t+T) = f(t)$ *for* $t > 0$ *then*

$$\mathcal{L}\left\{f(t)\right\}(s) = \frac{\int_0^T f(t)e^{-st}\,\mathrm{d}t}{1-e^{-sT}}$$

Theorem (Laplace Transform of a convolution)

Let $(f \star g)(t)$ *defined as:*

$$(f \star g)(t) = \int_0^t f(z)g(t-z)\,\mathrm{d}z$$

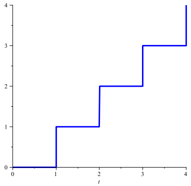
then

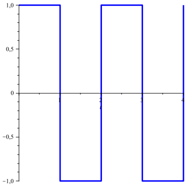
$$\mathcal{L}\left\{f \star g\right\}(s) = \widehat{f}(s)\,\widehat{g}(s)$$

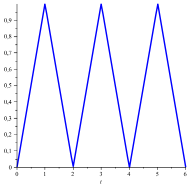
Table 3		
$\int_0^t f(z)\,\mathrm{d}z$	$\frac{1}{s}\widehat{f}(s)$	9
$f'(t)$	$s\widehat{f}(s) - f(0^+)$	10
$f''(t)$	$s^2\widehat{f}(s) - f'(0^+) - sf(0^+)$	11
$\frac{\mathrm{d}^n}{\mathrm{d}t^n} f(t)$	$s^n\widehat{f}(s) - \sum_{j=0}^{n-1} s^{n-j-1} f^{(j)}(0^+)$	12
$t^n f(t)$	$(-1)^n \frac{\mathrm{d}^n}{\mathrm{d}s^n}\widehat{f}(s)$	13
$(f \star g)(t)$	$\widehat{f}(s)\,\widehat{g}(s)$	14

Table 4		
$e^{at}\cos\omega t$	$\dfrac{s-a}{(s-a)^2+\omega^2}$	15
$e^{at}\sin\omega t$	$\dfrac{\omega}{(s-a)^2+\omega^2}$	16
$e^{at}\cosh\omega t$	$\dfrac{s-a}{(s-a)^2-\omega^2}$	17
$e^{at}\sinh\omega t$	$\dfrac{\omega}{(s-a)^2-\omega^2}$	18
$e^{at}t^n$	$\dfrac{n!}{(s-a)^{n+1}}$	19
$e^{\alpha t}-e^{\beta t}$	$\dfrac{\alpha-\beta}{(s-\alpha)(s-\beta)}$	20

1. $f(t) = \begin{cases} 0 & t < 0 \\ n & n \leq t < n+1 \end{cases}$
2. $g(t) = \begin{cases} 0 & t < 0 \\ +1 & 2n \leq t < 2n+1 \\ -1 & 2n+1 \leq t < 2n+2 \end{cases}$
3. $h(t) = \begin{cases} 0 & t < 0 \\ t-2n & 2n \leq t < 2n+1 \\ 2n+2-t & 2n+1 \leq t < 2n+2 \end{cases}$

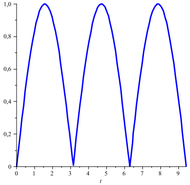


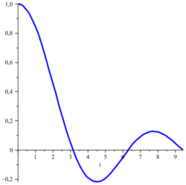


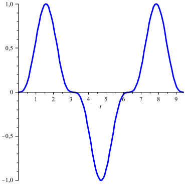


$$\widehat{f}(s) = \frac{1}{(e^s - 1)\, s}; \qquad \widehat{g}(s) = \frac{1}{s}\frac{e^s - 1}{e^s + 1}; \qquad \widehat{h}(s) = \frac{1}{s^2}\frac{e^s - 1}{e^s + 1}$$

1. $f(t) = |\sin(t)|$
2. $g(t) = \dfrac{\sin(t)}{t}$
3. $h(t) = \sin(t)^3$







$$\widehat{f}(s) = \frac{1}{1+s^2}\frac{e^{\pi s}+1}{e^{\pi s}-1}\ ; \qquad \widehat{g}(s) = \arctan(s);$$

$$\widehat{h}(s) = \frac{6}{(s^2+1)(s^2+9)}$$

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