Soluzioni del compito di Calcolo Numerico del 25 febbraio 2005

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Minimi quadrati

```
> restart:
  with(LinearAlgebra) :
```

Tabella dei punti

```
> sol := x -> 1+2*x+3*x^2+4*x^3 ;
```

$$sol := x \to 1 + 2\,x + 3\,x^2 + 4\,x^3$$

```
> X := [-2,-2,-1,0,1,2,3,1];
  Y := [seq(sol(X[i]),i=1..nops(X))];
```

$$X := [-2, -2, -1, 0, 1, 2, 3, 1]$$

$$Y := [-23, -23, -2, 1, 10, 49, 142, 10]$$

```
> n := nops(X) ;
```

$$n := 8$$

```
> SX   := add(X[i],i=1..n) ;
  SX2  := add(X[i]^2,i=1..n) ;
  SX3  := add(X[i]^3,i=1..n) ;
  SX4  := add(X[i]^4,i=1..n) ;
  SX5  := add(X[i]^5,i=1..n) ;
  SX6  := add(X[i]^6,i=1..n) ;
  SY   := add(Y[i],i=1..n) ;
  SXY  := add(X[i]*Y[i],i=1..n) ;
  SX2Y := add(X[i]^2*Y[i],i=1..n) ;
  SX3Y := add(X[i]^3*Y[i],i=1..n) ;
```

$$SX := 2$$

$$SX2 := 24$$

$$SX3 := 20$$

$$SX4 := 132$$

$$SX5 := 212$$

$$SX6 := 924$$

$$SY := 164$$

$$SXY := 638$$

$$SX2Y := 1308$$

$$SX3Y := 4616$$

```
> A :=
  <<n,SX,SX2,SX3>|<SX,SX2,SX3,SX4>|<SX2,SX3,SX4,SX5>|<SX3,SX4,SX5,SX6>>
  ;
```

$$A := \begin{bmatrix} 8 & 2 & 24 & 20 \\ 2 & 24 & 20 & 132 \\ 24 & 20 & 132 & 212 \\ 20 & 132 & 212 & 924 \end{bmatrix}$$

```
> b := <SY,SXY,SX2Y,SX3Y> ;
```

$$b := \begin{bmatrix} 164 \\ 638 \\ 1308 \\ 4616 \end{bmatrix}$$

```
> res := LinearSolve(A,b) ;
  evalf(res);
```

$$res := \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 1. \\ 2. \\ 3. \\ 4. \end{bmatrix}$$

Soluzione

```
> x := 'x' :
  p := 'p' :
> p := res[1]+ x * res[2] + x^2 * res[3]+ x^4 * res[4] ;
```

$$p := 1 + 2\,x + 3\,x^2 + 4\,x^4$$

```
> evalf(p,4) ;
```

$$1. + 2.\,x + 3.\,x^2 + 4.\,x^4$$

Differenze divise

```
> restart:
  with(LinearAlgebra) :
```

Soluzione del problema di interpolazione

```
> psol := x -> x-6*x^2;
```

$$psol := x \to x - 6\,x^2$$

Punti di interpolazione:

```
> X := [-2,0,1,2,5];
  Y := [seq(psol(X[i]),i=1..5)];
```

$$X := [-2, 0, 1, 2, 5]$$

$$Y := [-26, 0, -5, -22, -145]$$

Polinomio interpolante:

```
> interp( X, Y, 'z');
```

$$-6\,z^2 + z$$

Costruzione delle differenze divise di ordine 0

```
> f1 := Y[1];
  f2 := Y[2];
  f3 := Y[3];
  f4 := Y[4];
  f5 := Y[5];
```

$$f1 := -26$$

$$f2 := 0$$

$$f3 := -5$$

$$f4 := -22$$

$$f5 := -145$$

Differenze divise

```
> f12 := (f2-f1)/(X[2]-X[1]);
  f23 := (f3-f2)/(X[3]-X[2]);
  f34 := (f4-f3)/(X[4]-X[3]);
  f45 := (f5-f4)/(X[5]-X[4]);
```

$$f12 := 13$$

$$f23 := -5$$

$$f34 := -17$$

$$f45 := -41$$

Differenze divise seconde

```
> f123 := (f23-f12)/(X[3]-X[1]);
  f234 := (f34-f23)/(X[4]-X[2]);
  f345 := (f45-f34)/(X[5]-X[3]);
```

$$f123 := -6$$

$$f234 := -6$$

$$f345 := -6$$

Differenze divise terze

```
> f1234 := (f234-f123)/(X[4]-X[1]);
  f2345 := (f345-f234)/(X[5]-X[2]);
```

$$f1234 := 0$$

$$f2345 := 0$$

Differenze divise quarte

```
> f12345 := (f2345-f1234)/(X[5]-X[1]);
```

$$f12345 := 0$$

Polinomi della base

```
> w0 := 1 ;
  w1 := x-X[1] ;
  w2 := expand(w1 * ( x - X[2])) ;
  w3 := expand(w2 * ( x - X[3])) ;
  w4 := expand(w3 * ( x - X[4])) ;
```

$$w0 := 1$$

$$w1 := x + 2$$

$$w2 := x^2 + 2\,x$$

$$w3 := x^3 + x^2 - 2\,x$$

$$w4 := x^4 - x^3 - 4\,x^2 + 4\,x$$

Polinomio interpolante

```
> p := f1*w0 + f12 * w1 + f123 * w2 + f1234 * w3 + f12345 * w4 ;
```

$$p := x - 6\,x^2$$

Jacobi+Seidel

```
> restart;
  with(LinearAlgebra):
```

Matrice del sistema

```
> L  := <<0,-1,-1>|<0,0,-1>|<0,0,0>>;
  DG := <<6,0,0>|<0,3,0>|<0,0,2>>;
  U  := <<0,0,0>|<-2,0,0>|<-2,-2,0>>;
  A  := DG+L+U ;
```

$$L := \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ -1 & -1 & 0 \end{bmatrix}$$

$$DG := \begin{bmatrix} 6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$U := \begin{bmatrix} 0 & -2 & -2 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$A := \begin{bmatrix} 6 & -2 & -2 \\ -1 & 3 & -2 \\ -1 & -1 & 2 \end{bmatrix}$$

Termine noto

```
> b := <1,-1,1>;
```

$$b := \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

libera x, y, z

```
> x := 'x' ;
  y := 'y' ;
  z := 'z' ;
```

$$x := x$$

$$y := y$$

$$z := z$$

Metodo Iterativo

```
> iter := (DG+L).<x[k+1],y[k+1],z[k+1]> + U.<x[k],y[k],z[k]> - b ;
```

$$iter := \begin{bmatrix} 6\,x_{k+1} - 2\,y_k - 2\,z_k - 1 \\ -x_{k+1} + 3\,y_{k+1} - 2\,z_k + 1 \\ -x_{k+1} - y_{k+1} + 2\,z_{k+1} - 1 \end{bmatrix}$$

```
> isolate(iter[1]=0,x[k+1]);
  isolate(iter[2]=0,y[k+1]);
  isolate(iter[3]=0,z[k+1]);
```

$$x_{k+1} = \frac{1}{3}\,y_k + \frac{1}{3}\,z_k + \frac{1}{6}$$

$$y_{k+1} = \frac{1}{3}\,x_{k+1} + \frac{2}{3}\,z_k - \frac{1}{3}$$

$$z_{k+1} = \frac{1}{2}\,x_{k+1} + \frac{1}{2}\,y_{k+1} + \frac{1}{2}$$

Matrice di iterazione -(D+L)^(-1)U

```
> MI := -(DG+L)^(-1).U;
```

$$MI := \begin{bmatrix} 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{9} & \frac{7}{9} \\ 0 & \frac{2}{9} & \frac{5}{9} \end{bmatrix}$$

Polinomio caratteristico della matrice di iterazione

```
> chpoly := CharacteristicPolynomial(MI,z);
```

$$chpoly := z^3 - \frac{2}{3} z^2 - \frac{1}{9} z$$

```
> res := solve(chpoly,z);
```

$$res := 0, \frac{1}{3} + \frac{1}{3}\sqrt{2}, \frac{1}{3} - \frac{1}{3}\sqrt{2}$$

Calcolo il raggio spettrale

```
> seq(abs(res[i]),i=1..3);
  evalf(%);
```

$$0, \frac{1}{3} + \frac{1}{3}\sqrt{2}, -\frac{1}{3} + \frac{1}{3}\sqrt{2}$$

$$0., 0.8047378539, 0.1380711873$$

Raggio spettrale = 1/2 < 1 ==> il metodo converge.

Faccio alcune iterate:

```
> P0 := <2,1,2> ;
```

$$P0 := \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$

```
> P1 := (DG+L)^(-1).(b-U.P0):
  P1, evalf(P1) ;
```

$$\begin{bmatrix} \frac{7}{6} \\ \frac{25}{18} \\ \frac{16}{9} \end{bmatrix}, \begin{bmatrix} 1.166666667 \\ 1.388888889 \\ 1.777777778 \end{bmatrix}$$

```
> P2 := (DG+L)^(-1).(b-U.P1) :
  P2, evalf(P2) ;
```

$$\begin{bmatrix} \frac{11}{9} \\ \frac{34}{27} \\ \frac{47}{27} \end{bmatrix}, \begin{bmatrix} 1.222222222 \\ 1.259259259 \\ 1.740740741 \end{bmatrix}$$

Calcolo i residui

```
> R1 := b - A . P1 :
  R2 := b - A . P2 :
  R1, evalf(R1) ;
  R2, evalf(R2) ;
```

$$\begin{bmatrix} \frac{1}{3} \\ \frac{-4}{9} \\ 0 \end{bmatrix}, \begin{bmatrix} 0.3333333333 \\ -0.4444444444 \\ 0. \end{bmatrix}$$

$$\begin{bmatrix} \frac{-1}{3} \\ \frac{-2}{27} \\ 0 \end{bmatrix}, \begin{bmatrix} -0.3333333333 \\ -0.07407407407 \\ 0. \end{bmatrix}$$

Norma infinito

```
> max(seq(abs(R1[i]),i=1..3));
  max(seq(abs(R2[i]),i=1..3));
  evalf(%%),evalf(%) ;
```

$$\frac{4}{9}$$

$$\frac{1}{3}$$

$$0.4444444444, 0.3333333333$$

Norma 1

```
> add(abs(R1[i]),i=1..3);
  add(abs(R2[i]),i=1..3);
  evalf(%%),evalf(%) ;
```

$$\frac{7}{9}$$

$$\frac{11}{27}$$

$$0.7777777778, 0.4074074074$$

Norma 2

```
> sqrt(add(abs(R1[i])^2,i=1..3));
  sqrt(add(abs(R2[i])^2,i=1..3));
  evalf(%%),evalf(%) ;
```

$$\frac{5}{9}$$

$$\frac{1}{27}\sqrt{85}$$

$$0.5555555556, 0.3414646095$$

BVP

```
> restart:
  with(LinearAlgebra) :
```

Definizione del problema

```
> p := x -> 1-2*x^2 ;
  q := x -> -x^2 ;
  r := x -> x - 4 ;
  xa, xb := -2, 2 ;
  ya, yb := 0, 14 ;
```

$$p := x \to 1 - 2\,x^2$$

$$q := x \to -x^2$$

$$r := x \to x - 4$$

$$xa, xb := -2, 2$$

$$ya, yb := 0, 14$$

Differenze finite:

```
> n := 4 ;
  h := (xb-xa) / n ;
  x[0] := xa ;
  x[1] := xa + h ;
  x[2] := xa + 2*h ;
  x[3] := xa + 3*h ;
  x[4] := xa + 4*h ;
```

$$n := 4$$

$$h := 1$$

$$x_0 := -2$$

$$x_1 := -1$$

$$x_2 := 0$$

$$x_3 := 1$$

$$x_4 := 2$$

```
> eq := k -> (y[k+1]-2*y[k]+y[k-1])/h^2 +
```

```
            max(p(x[k]),0) * (y[k+1]-y[k])/h +
            min(p(x[k]),0) * (y[k]-y[k-1])/h +
            q(x[k]) * y[k] - r(x[k]) ;
```

$$eq := k \to \frac{y_{k+1} - 2\,y_k + y_{k-1}}{h^2} + \frac{\max(p(x_k), 0)\,(y_{k+1} - y_k)}{h} + \frac{\min(p(x_k), 0)\,(y_k - y_{k-1})}{h} + q(x_k)\,y_k - r(x_k$$

Equazioni risultanti

```
> eq1 := eq(1) ;
  eq2 := eq(2) ;
  eq3 := eq(3) ;
```

$$eq1 := y_2 - 4\,y_1 + 2\,y_0 + 5$$

$$eq2 := 2\,y_3 - 3\,y_2 + y_1 + 4$$

$$eq3 := y_4 - 4\,y_3 + 2\,y_2 + 3$$

Estraggo il sistema lineare dalle equazioni

```
> A := linalg[genmatrix]([eq1,eq2,eq3],[y[1],y[2],y[3]],'b'):
  A := convert(A,Matrix);
```

$$A := \begin{bmatrix} -4 & 1 & 0 \\ 1 & -3 & 2 \\ 0 & 2 & -4 \end{bmatrix}$$

```
> b := Transpose(convert(b,Vector));
```

$$b := \begin{bmatrix} -2\,y_0 - 5 \\ -4 \\ -y_4 - 3 \end{bmatrix}$$

Sostituisco le condizioni al contorno

```
> b := subs(y[0]=ya, y[4]=yb, b) ;
```

$$b := \begin{bmatrix} -5 \\ -4 \\ -17 \end{bmatrix}$$

Risolvo il sistema lineare

```
> res := LinearSolve(A,b) ;
  evalf(res) ;
```

$$res := \begin{bmatrix} \frac{45}{14} \\ \frac{55}{7} \\ \frac{229}{28} \end{bmatrix}$$

$$\begin{bmatrix} 3.214285714 \\ 7.857142857 \\ 8.178571429 \end{bmatrix}$$

>