# Numerical Methods for Dynamic System and Control del 17/1/2012

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## 1

Solve the following recurrence

$$
\begin{aligned}
x_{k+1} &= x_k + y_k - k + 1, & x_0 &= 0 \\
y_{k+1} &= y_k + z_k + 1 - k^2 + k, & y_0 &= 0 \\
z_{k+1} &= x_k + y_k + z_k, & z_0 &= 0
\end{aligned}
$$

$\mathcal{Z}-$transform:

$$
\begin{aligned}
(\zeta - 1)x(\zeta) - y(\zeta) &= -\frac{\zeta}{(\zeta-1)^2} + \frac{\zeta}{\zeta - 1}, \\
(\zeta - 1)y(\zeta) - z(\zeta) &= \frac{\zeta}{\zeta - 1} - \frac{\zeta(\zeta+1)}{(\zeta-1)^3} + \frac{\zeta}{(\zeta-1)^2} \\
(\zeta - 1)z(\zeta) - y(\zeta) - x(\zeta) &= 0
\end{aligned}
$$

Solution in $\mathcal{Z}$:

$$
\begin{aligned}
x(\zeta) &= \frac{\zeta}{(\zeta-1)^2}, \\
y(\zeta) &= \frac{\zeta}{(\zeta-1)^2}, \\
z(\zeta) &= 2\frac{\zeta}{(\zeta-1)^3},
\end{aligned}
$$

Solution in $k$

$$
\begin{aligned}
x_k &= k, \\
y_k &= k, \\
z_k &= k(k-1).
\end{aligned}
$$

# 2 Livello difficoltà 2

Given the following system of ordinary differential equations

$$\begin{aligned} y''(t) + x''(t) &= 2 + (2+t)e^t \\ x''(t) + x'(t) - y'(t) &= (3+2t)e^t - 1 - 2t \end{aligned}$$

$$x(0) = 0, \quad y(0) = 0, \quad x'(0) = 1, \quad y'(0) = 1,$$

Compute:

The Laplace transform:

$$\begin{cases} s^2 y(s) - 2 + s^2 x(s) &= \dfrac{4s^2 - 5s + 2}{s(s-1)^2} \\ s(s+1)x(s) - 1 - s y(s) &= \dfrac{2s^3 - s^2 + 3s - 2}{(s-1)^2 s^2} \end{cases}$$

Solution in $s$:

$$\begin{cases} x(s) &= \dfrac{1}{(s-1)^2} \\ y(s) &= \dfrac{s+2}{s^3} \end{cases}$$

Solution in $t$:

$$\begin{cases} x(t) &= t\,e^t \\ y(t) &= t(t+1) \end{cases}$$

# 3

Given the following system of ODE:

$$\begin{cases} x'(t) + y(t) = 1 + e^t \\ y''(t) - y(t) = 0 \\ x(0) = 0, \qquad y(0) = 1, \qquad y'(0) = A, \end{cases}$$

Compute constant $A$ in such a way $x(1) = 1$.

Laplace transform:

$$\begin{cases} s\,x(s) + y(s) &= \dfrac{2s-1}{s(s-1)} \\ s^2 y(s) - A - s - y(s) &= 0 \end{cases}$$

Solution in $s$:

$$\begin{cases} x(s) &= \dfrac{1+s+s^2-s\,A}{s^2(s^2-1)}, \\ y(s) &= \dfrac{A+s}{s^2-1} \end{cases}$$

Solution in $t$:

$$\begin{cases} x(t) &= t &= t - 1 + A + (1-A)\cosh(t), \\ y(t) &= e^t &= \cosh(t) + A\sinh(t), \end{cases}$$

Constant $A$

$$A = 1$$

# 4

Study the following constrained minimization problem.

$$\text{minimize} \quad f(x,y) = x + y + x\,y, \qquad x \geq y^2 - 1, \qquad y \geq x^2 - 1,$$

Notice that: $(a+b\sqrt{5})(c+d\sqrt{5}) = ac + 5bd + (ad+bc)\sqrt{5}$
and $z^3 - 2z - 1 = 0$ has solutions $z = -1, \dfrac{1 \pm \sqrt{5}}{2}$.

KKT system of first order condition:

$$\begin{cases} 1 + y - \mu_1 + 2\mu_2 x &= 0 \\ 1 + x + 2\mu_1 y - \mu_2 &= 0 \\ \mu_1(x - y^2 + 1) &= 0 \\ \mu_2(y - x^2 + 1) &= 0 \end{cases}$$

Solutions of KKT system:

$$\begin{cases} x = -1, \quad y = 0, \quad \mu_1 = 1, \quad \mu_2 = 0 \\ x = 0, \quad y = -1, \quad \mu_1 = 0, \quad \mu_2 = 1 \\ x = -\frac{5}{9}, \quad y = -\frac{2}{3}, \quad \mu_1 = \frac{1}{3}, \quad \mu_2 = 0 \\ x = -\frac{2}{3}, \quad y = -\frac{5}{9}, \quad \mu_1 = 0, \quad \mu_2 = \frac{1}{3}, \\ x = y = \frac{1}{2}(1-\sqrt{5}), \quad \mu_1 = \mu_2 = -\frac{1}{2} + \frac{3\sqrt{5}}{10} \end{cases}$$

Discussion of the stationary point: $x = -\frac{5}{9}, \quad y = -\frac{2}{3}, \quad \mu_1 = \frac{1}{3}, \quad \mu_2 = 0.$

## 5

Compute matrix exponential of the following matrix:

$$\boldsymbol{A} = \begin{pmatrix} 5 & 1/2 & 1 \\ 0 & 2 & 0 \\ -3 & 1/2 & 1 \end{pmatrix}$$

Eigenvalues

$\lambda_1 = 4, \qquad \lambda_2 = 2, \qquad \lambda_3 = 2,$

Matrix exponential

$$e^{\boldsymbol{A}} = \begin{pmatrix} 78.21 & 19.91 & 23.61 \\ 0 & 7.389 & 0 \\ -70.82 & -12.52 & -16.22 \end{pmatrix}$$

## 6

Given the following function

$$f(x) = \begin{cases} 0 & \text{for } -\pi \le x < 0 \\ \sin x & \text{for } 0 \le x < \pi \end{cases}$$

defined for $x \in (-\pi, \pi)$ and extended periodically. Compute the coefficients of Fourier series:

$$a_0 = \frac{2}{\pi}$$

$$a_k = -\frac{1 + \cos(\pi k)}{(k^2 - 1)\pi} = -\frac{1 + (-1)^k}{(k^2 - 1)\pi} = \begin{cases} 0 & \text{for } k = 2m \\ -\dfrac{2}{(4\,m^2 - 1)\pi} & \text{for } k = 2m + 1 \end{cases}$$

$$b_k = \begin{cases} 1/2 & \text{for } k = 1 \\ 0 & \text{for } k > 1 \end{cases}$$

## 7

Solve the following optimal control problem

$$\begin{cases} \text{Minimize:} \quad \displaystyle\int_0^2 u(t)^2 - 2x(t)\,\mathrm{d}t \\ x'(t) = 1 - 2\,u(t) \\ x(0) = 1, \quad x(2) = 0 \end{cases}$$

| The Boundary Value Problem (BVP): |
| --- |
| The optimal control $u(t)$: |
| The solution of the BVP: |