Solution of the exam

Numerical Methods for Dynamical Systems and Control

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1 Exercise 7

Solve the following optimal control problem (OCP).

$$\min J = \min \int_0^2 u(t)^2 - 2x(t)\ \mathrm{d}t \qquad \text{with} \qquad x'(t) = 1 - 2u(t) \qquad x(0) = 1,\ x(2) = 0.$$

1.1 Solution with variation calculus

To solve the problem, one can use calculus of variation in order to find a maximum of the functional J, so

$$\min J = \max -J = \max \int_0^2 -u(t)^2 + 2x(t)\ \mathrm{d}t.$$

The Lagrangian function is given by

$$\mathcal{L}(x, u, \lambda, \mu_1, \mu_2) = \int_0^2 -u^2 + 2x - \lambda(x' - 1 + 2u)\ \mathrm{d}t - \mu_1(x(0) - 1) - \mu_2(x(2) - 0)$$

Now performing the first variation of $\mathcal{L}$ yields

$$\delta\mathcal{L} = \int_0^2 -2u\delta_u + 2\delta_x - \lambda(\delta_{x'} + 2\delta_u) - \delta_\lambda(x' - 1 + 2u)\ \mathrm{d}t$$
$$-\mu_1\delta_{x(0)} - \delta_{\mu_1}(x(0) - 1) - \mu_2\delta_{x(2)} - \delta_{\mu_2}(x(2) - 0).$$

To simplify the variation $\delta_{x'}$ it is enough to derive $\lambda\delta_x$,

$$(\lambda\delta_x)' = \lambda'\delta_x + \lambda\delta_{x'} \implies \lambda\delta_{x'} = (\lambda\delta_x)' - \lambda'\delta_x,$$

so the previous expression becomes

$$\delta\mathcal{L} = \int_0^2 -2u\delta_u + 2\delta_x + \lambda'\delta_x - 2\lambda\delta_u - \delta_\lambda(x' - 1 + 2u)\ \mathrm{d}t$$
$$-\lambda(2)\delta_{x(2)} + \lambda(0)\delta_{x(0)} - \mu_1\delta_{x(0)} - \delta_{\mu_1}(x(0) - 1) - \mu_2\delta_{x(2)} - \delta_{\mu_2}(x(2) - 0).$$

Collecting the expression of each variation gives the associated boundary value problem.

$$\begin{aligned} &\delta_u : \; -2u - 2\lambda && \delta_x : \; 2 + \lambda' && \delta_\lambda : \; x' - 1 + 2u \\ &\delta_{x(0)} : \; \lambda(0) - \mu_1 && \delta_{x(2)} : \; -\lambda(2) - \mu_2 && \\ &\delta_{\mu_1} : \; x(0) - 1 && \delta_{\mu_2} : \; x(2) && \end{aligned}$$

From the variation δ_x one can solve the multiplier λ, in facts

$$\lambda' = -2 \implies \lambda(t) = -2t + c \qquad c \in \mathbb{R}.$$

From variation δ_u one can solve the optimal control u,

$$-2u - 2\lambda = -2u - 2(-2t + c) = 0 \implies u = 2t - c.$$

From the differential equation given by the variation of the multiplier λ

$$x' = 1 - 2u \implies x' = 1 - 2t + c \implies x(t) = -t^2 + (c+1)t + d \qquad d \in \mathbb{R}.$$

Now the constants c, d can be evaluated using the boundary condition $x(0) = 1$ and $x(2) = 0$.

$$\begin{aligned} x(0) &= d = 1 && \implies d = 1 \\ x(2) &= -4 + 2c + 2 + 1 = 0 && \implies c = \frac{1}{2}. \end{aligned}$$

In conclusion the optimal control is $u(t) = 2t - \frac{1}{2}$ and the associated trajectory or status is $x(t) = -t^2 + \frac{3}{2}t + 1$. The extremal value of the functional is therefore

$$\begin{aligned} \max \int_0^2 -u(t)^2 + 2x(t) \, \mathrm{d}t &= \int_0^2 -\left(2t - \frac{1}{2}\right)^2 + 2\left(-t^2 + \frac{3}{2}t + 1\right) \, \mathrm{d}t \\ &= \int_0^2 -6t^2 + 5t + 7/4 \, \mathrm{d}t \\ &= -2t^3 + \frac{5}{2}t^2 + \frac{7}{4}t \Big|_0^2 \\ &= -\frac{5}{2}. \end{aligned}$$

If one tries another function that satisfies the required constraints, for example $x(t) = (t-2)(t-\frac{1}{2})$ and the associated control $u(t) = 2t - \frac{5}{2}$, that the functional $-J$ gives $-\frac{23}{6} \approx -3.833 < -2.5$.
It can be showed, for example using the Hamiltonian function or with the second variation, that it is a maximum point[1], so for the original problem it will be a minimum.

1 $\dfrac{\partial^2 \mathcal{H}}{\partial u^2} = -2 < 0$