Numerical Methods for Dynamic System and Control del 7/2/2012

SURNAME | NAME | MAT. NUMBER

Signature ______________________

1

Solve the following optimal control problem

$$\begin{cases} \text{Minimize:} \quad \displaystyle\int_{-1}^{1} t x(t)\,\mathrm{d}t \\ x'(t) = u(t) \\ x(-1) = 0, \quad -1 \leq u(t) \leq 1 \end{cases}$$

The Boundary Value Problem (BVP): $\begin{cases} x'(t) = u(t), \\ \lambda'(t) = -t, \\ x(-1) = 0, \\ \lambda(1) = 0 \\ u(t) = \arg\min_{u \in [-1,1]} \{ t x(t) + \lambda(t) u \} \end{cases}$

The optimal control $u(t)$: $u(t) = -1$

The solution of the BVP: $\begin{cases} x(t) = -t - 1, \\ \lambda(t) = (1 - t^2)/2 \end{cases}$

2

Study the following constrained minimization problem.

$$\text{minimize} \quad f(x,y) = x\,y, \qquad x^2 + y^2 \le 1, \qquad x^2 \le y,$$

KKT system of first order condition:

$$\begin{cases} y + 2\mu_1 x + 2\mu_2 x &= 0 \\ x + 2\mu_1 y - \mu_2 &= 0 \\ \mu_1(1 - x^2 - y^2) &= 0 \\ \mu_2(y - x^2) &= 0 \end{cases}$$

Solutions of KKT system:

$$\begin{cases} x = y = \mu_1 = \mu_2 = 0 \\ x = -\dfrac{\sqrt{2}}{2}, \quad y = \dfrac{\sqrt{2}}{2}, \quad \mu_1 = \dfrac{1}{2}, \quad \mu_2 = 0 \end{cases}$$

Discussion of the stationary point: $x = -\dfrac{\sqrt{2}}{2}, \quad y = \dfrac{\sqrt{2}}{2}, \quad \mu_1 = \dfrac{1}{2}, \quad \mu_2 = 0.$

3

Solve the following recurrence

$$\begin{aligned} x_{k+1} &= x_k + 2y_k - 1 - 2^{k+1}, & x_0 &= 0 \\ y_{k+1} &= 2y_k + z_k - k(k-1), & y_0 &= 1 \\ z_{k+1} &= z_k + y_k + x_k + 3k - 2^k, & z_0 &= 0 \end{aligned}$$

$\mathcal{Z}-$transform:

$$\begin{aligned} (\zeta-1)x(\zeta) - 2y(\zeta) &= -\frac{\zeta}{\zeta-1} - \frac{2\zeta}{\zeta-2} \\ (\zeta-2)y(\zeta) - \zeta - z(\zeta) &= -\frac{\zeta(\zeta+1)}{(\zeta-1)^3} + \frac{\zeta}{(\zeta-1)^2} = -\frac{\zeta}{(\zeta-1)^3} \\ (\zeta-1)z(\zeta) - y(\zeta) - x(\zeta) &= \frac{3\zeta}{(\zeta-1)^2} - \frac{\zeta}{\zeta-2} \end{aligned}$$

Solution in $\mathcal{Z}$:

$$\begin{aligned} x(\zeta) &= -\frac{\zeta}{(\zeta-1)^2}, \\ y(\zeta) &= \frac{\zeta}{\zeta-2}, \\ z(\zeta) &= \frac{2\zeta}{(\zeta-1)^3}, \end{aligned}$$

Solution in k

$$\begin{aligned} x_k &= -k, \\ y_k &= 2^k, \\ z_k &= k(k-1). \end{aligned}$$

4 Livello difficoltà 2

Given the following system of ordinary differential equations

$$
\begin{aligned}
y''(t) + x''(t) &= -4\cos(2t) + (2+t)e^t \\
x''(t) + x'(t) - y'(t) &= (3+2t)e^t + 2\sin(2t) \\
x(0) = 0, \quad y(0) = 1, &\quad x'(0) = 1, \quad y'(0) = 0,
\end{aligned}
$$

Compute:

The Laplace transform:

$$
\begin{cases}
s^2 y(s) - 1 - s + s^2 x(s) &= -\dfrac{2s^3 - 7s^2 - 4s + 4}{(s^2+4)(s-1)^2} \\
s(s+1)x(s) - s\,y(s) &= \dfrac{s(3s^2+3s+4)}{(s^2+4)(s-1)^2}
\end{cases}
$$

Solution in s:

$$
\begin{cases}
x(s) &= \dfrac{1}{(s-1)^2} \\
y(s) &= \dfrac{s}{s^2+4}
\end{cases}
$$

Solution in t:

$$
\begin{cases}
x(t) &= t\,e^t \\
y(t) &= \cos(2t)
\end{cases}
$$

5

Given the following system of ODE:

$$\begin{cases} x'(t) + 2x(t) = e^{-t} \\ \\ y''(t) - x(t) = e^{t} - e^{-t} \\ \\ x(0) = 1, \qquad y(0) = 1, \qquad y'(0) = A, \\ \end{cases}$$

Compute constant A in such a way $y(1) = e^1 = e$.

Laplace transform:

$$\begin{cases} s\,x(s) - 1 + 2x(s) &= \dfrac{1}{1+s} \\ s^2 y(s) - A - s - x(s) &= \dfrac{2}{s^2-1} \end{cases}$$

Solution in s:

$$\begin{cases} x(s) &= \dfrac{1}{1+s}, \\ y(s) &= \dfrac{-s + s^2 - A + s\,A + 1}{s^2(s-1)} \end{cases}$$

Solution in t:

$$\begin{cases} x(t) &= \quad = \; e^{-t} \\ y(t) &= \quad = \; e^{t} + (A-1)t \end{cases}$$

Constant A

$$A \; = \; 1$$

6

Compute matrix exponential of the following matrix:

$$\boldsymbol{A} = \begin{pmatrix} 1 & 2 & -2 \\ 0 & 2 & 0 \\ 1 & 0 & 4 \end{pmatrix}$$

Eigenvalues

$\lambda_1 = 3, \qquad \lambda_2 = 2, \qquad \lambda_3 = 2,$

Matrix exponential

$$e^{\boldsymbol{A}} = \begin{pmatrix} -5.31 & 4.15 & -25.40 \\ 0 & 7.389 & 0 \\ 12.70 & 10.62 & 32.79 \end{pmatrix}$$

7

Given the following function

$$f(x) = \begin{cases} -x & \text{for } -\pi \le x < 0 \\ \pi/2 & \text{for } 0 \le x < \pi \end{cases}$$

defined for $x \in (-\pi, \pi)$ and extended periodically. Compute the coefficients of Fourier series:

$a_0 = \pi$

$$a_k = \frac{(-1)^k - 1}{k^2\pi} 4 = \begin{cases} 0 & \text{for } k = 2m \\ -\dfrac{2}{(2\,m+1)^2\pi} & \text{for } k = 2m+1 \end{cases}$$

$$b_k = \frac{(-1)^k + 1}{2k} = \begin{cases} \dfrac{1}{2m} & \text{for } k = 2m \\ 0 & \text{for } k = 2m+1 \end{cases}$$