Numerical Methods for Dynamic System and Control del 5/7/2012

Surname | | Name | | Mat. number | |

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Solve the following optimal control problem

$$\begin{cases} \text{Minimize:} \quad \int_0^2 x(t)\,\mathrm{d}t \\ x'(t) = x(t) + u(t) \\ x(0) = 0, \quad 0 \leq u(t) \leq 1 \end{cases}$$

The Boundary Value Problem (BVP):

The optimal control $u(t)$:

The solution of the BVP:

2

Study the following constrained minimization problem.

$$\text{minimize} \quad f(x,y) = x\,y + x^2, \qquad x \geq y^2, \qquad x^2 + y \leq 1,$$

KKT system of first order condition:

$$\begin{cases} y + 2x - \mu_1 + 2\mu_2 x &= 0 \\ x + 2\mu_1 y + \mu_2 &= 0 \\ \mu_1(x - y^2) &= 0 \\ \mu_2(1 - x^2 - y) &= 0 \end{cases}$$

Solutions of KKT system:

$$\begin{cases} x = y = \mu_1 = \mu_2 = 0 \\ x = \dfrac{9}{16}, \quad y = -\dfrac{3}{4}, \quad \mu_1 = \dfrac{3}{8}, \quad \mu_2 = 0 \end{cases}$$

Discussion of the stationary point: $x = \dfrac{9}{16}, \quad y = -\dfrac{3}{4}, \quad \mu_1 = \dfrac{3}{8}, \quad \mu_2 = 0.$

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Solve the following recurrence

$$x_{k+1} = x_k + 2y_k - 1, \qquad x_0 = 0$$
$$y_{k+1} = y_k + 2z_k - 2k(k-1), \qquad y_0 = 1$$
$$z_{k+1} = z_k + y_k + x_k + k - 1, \qquad z_0 = 0$$

$\mathcal{Z}$−transform:

$$(\zeta - 1)x(\zeta) - 2y(\zeta) = -\frac{\zeta}{\zeta - 1}$$

$$(\zeta - 1)y(\zeta) - \zeta - 2z(\zeta) = -\frac{2\zeta(\zeta+1)}{(\zeta-1)^3} + \frac{2\zeta}{(\zeta-1)^2}$$

$$(\zeta - 1)z(\zeta) - y(\zeta) - x(\zeta) = \frac{\zeta}{(\zeta-1)^2} - \frac{\zeta}{\zeta - 1}$$

Solution in $\mathcal{Z}$:

$$x(\zeta) = \frac{\zeta}{(\zeta-1)^2},$$

$$y(\zeta) = \frac{\zeta}{\zeta-1},$$

$$z(\zeta) = \frac{2 * \zeta}{(\zeta-1)^3},$$

Solution in k

$$x_k = k,$$

$$y_k = 1,$$

$$z_k = k(k-1).$$

4 Livello difficoltà 2

Given the following system of ordinary differential equations

$$y''(t) + x''(t) = -2\sin(t)e^t$$
$$x'(t) - y'(t) = (\cos(t) - \sin(t))e^t - 1$$
$$x(0) = 1, \quad y(0) = 0, \quad x'(0) = 1, \quad y'(0) = 1,$$

Compute:

The Laplace transform:

$$\begin{cases} s^2 y(s) - 2 + s^2 x(s) - s &= -\dfrac{2}{(s-1)^2+1} \\ s\,x(s) - 1 - s\,y(s) &= -\dfrac{2}{s((s-1)^2+1)} \end{cases}$$

Solution in s:

$$\begin{cases} x(s) &= \dfrac{s-1}{(s-1)^2+1} \\ y(s) &= \dfrac{1}{s^2} \end{cases}$$

Solution in t:

$$\begin{cases} x(t) &= \cos(t)e^t \\ y(t) &= t \end{cases}$$

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Given the following system of ODE:

$$\begin{cases} y'(t) + x(t) = 2\cos(t) \\ y''(t) = -\sin(t) \\ x(0) = 1, \qquad y(0) = 0, \qquad y'(0) = A, \end{cases}$$

Compute constant A in such a way $y(2\pi) = 1$.

Laplace transform:

$$\begin{cases} s\,y(s) + x(s) &= \dfrac{2\,s}{s^2+1} \\ s^2 * y(s) - A &= -\dfrac{1}{s^2+1} \end{cases}$$

Solution in s:

$$\begin{cases} x(s) &= \dfrac{-s^2A - A + 2s^2 + 1}{s(s^2+1)}, \\ y(s) &= \dfrac{-1 + s^2A + A}{s^2(s^2+1)}, \end{cases}$$

Solution in t:

$$\begin{cases} x(t) &= \quad = \; 1 - A + \cos(t), \\ y(t) &= \quad = \; (A-1)t + \sin(t) \end{cases}$$

Constant A

$$A = 1$$

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Compute matrix exponential of the following matrix:

$$\boldsymbol{A} = \begin{pmatrix} -1 & -2 & -1 \\ 4 & 5 & 1 \\ 5 & 3 & 4 \end{pmatrix}$$

Eigenvalues

$\lambda_1 = 2, \qquad \lambda_2 = 3, \qquad \lambda_3 = 3,$

Matrix exponential

$$e^{\boldsymbol{A}} = \begin{pmatrix} -38.10 & -32.79 & -12.70 \\ 58.19 & 52.88 & 12.70 \\ 78.23 & 52.88 & 32.79 \end{pmatrix}$$

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Given the following function

$$f(x) = \begin{cases} \pi + x & \text{for } -\pi \le x < 0 \\ \pi - x & \text{for } 0 \le x < \pi \end{cases}$$

defined for $x \in (-\pi, \pi)$ and extended periodically. Compute the coefficients of Fourier series:

$a_0 = \pi$

$$a_k = 2\frac{1 - \cos(k\pi)}{k^2\pi} = 2\frac{1 - (-1)^k}{k^2\pi} \begin{cases} 0 & \text{for } k = 2m \\ \dfrac{4}{(2\,m+1)^2\pi} & \text{for } k = 2m+1 \end{cases}$$

$b_k = 0,$