Numerical Methods for Dynamic System and Control del 23/7/2012

Surname | | Name | | Mat. number | |

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Solve the following optimal control problem

$$\begin{cases} \text{Minimize:} \quad \displaystyle\int_{-1}^{1} x(t) + t\,u(t)\,\mathrm{d}t \\ x'(t) = u(t) \\ x(-1) = 0, \quad x(1) = 0, \quad -1 \leq u(t) \leq 1 \end{cases}$$

The Boundary Value Problem (BVP):

The optimal control $u(t)$:

The solution of the BVP:

2

Study the following constrained minimization problem.

$$\text{minimize} \quad f(x,y) = x\,y - y, \qquad x\,y + x + y \geq 0, \qquad y \leq x^2,$$

KKT system of first order condition:

$$\begin{cases} y - \mu_1(y+1) - 2\mu_2 x &= 0 \\ x - 1 - \mu_1(x+1) + \mu_2 &= 0 \\ \mu_1(xy + x + y) &= 0 \\ \mu_2(x^2 - y) &= 0 \end{cases}$$

Solutions of KKT system:

$$\begin{cases} x = y = \mu_1 = 0, \quad \mu_2 = 1 \\ x = 1, \quad y = \mu_1 = \mu_2 = 0 \\ x = \dfrac{2}{3}, \quad y = \dfrac{4}{9}, \quad \mu_1 = 0, \quad \mu_2 = \dfrac{1}{3} \\ x = -1 - \sqrt{2}, \quad y = -1 - \dfrac{\sqrt{2}}{2}, \quad \mu_1 = 1 + \sqrt{2}, \quad \mu_2 = 0 \end{cases}$$

Discussion of the stationary point: $x = \dfrac{2}{3}, \quad y = \dfrac{4}{9}, \quad \mu_1 = 0, \quad \mu_2 = \dfrac{1}{3}.$

3

Solve the following recurrence

$$
\begin{aligned}
x_{k+1} &= 2x_k + y_k - 1, & x_0 &= 0\\
y_{k+1} &= 2y_k + z_k - k + 1 - 2^k, & y_0 &= 0\\
z_{k+1} &= z_k + y_k + x_k + 2^k, & z_0 &= 1
\end{aligned}
$$

$\mathcal{Z}-$transform:

$$
\begin{aligned}
(\zeta-2)x(\zeta) - y(\zeta) &= -\frac{\zeta}{\zeta-1}\\
(\zeta-2)y(\zeta) - z(\zeta) &= -\frac{\zeta}{(\zeta-1)^2} + \frac{\zeta}{\zeta-1} - \frac{\zeta}{\zeta-2}\\
(\zeta-1)z(\zeta) - \zeta - y(\zeta) - x(\zeta) &= \frac{\zeta}{\zeta-2}
\end{aligned}
$$

Solution in $\mathcal{Z}$:

$$
\begin{aligned}
x(\zeta) &= -\frac{\zeta}{(\zeta-1)^2},\\
y(\zeta) &= \frac{\zeta}{(\zeta-1)^2},\\
z(\zeta) &= \frac{\zeta}{\zeta-2},
\end{aligned}
$$

Solution in k

$$
\begin{aligned}
x_k &= -k,\\
y_k &= k,\\
z_k &= 2^k.
\end{aligned}
$$

4

Given the following system of ordinary differential equations

$$y''(t) + x'(t) = (1+t)e^t - \cos(t)$$
$$x'(t) - y'(t) = (1+t)e^t + \sin(t)$$
$$x(0) = 0, \quad y(0) = 1, \quad y'(0) = 0,$$

Compute:

The Laplace transform:

$$\begin{cases} s^2 y(s) - s + s\,x(s) &= \dfrac{2s^2}{(s^2+1)(s-1)^2} \\ s\,x(s) - s\,y(s) + 1 &= \dfrac{s^3+s^2-s+1}{(s^2+1)(s-1)^2} \end{cases}$$

Solution in s:

$$\begin{cases} x(s) &= \dfrac{1}{(s-1)^2} \\ y(s) &= \dfrac{s}{s^2+1} \end{cases}$$

Solution in t:

$$\begin{cases} x(t) &= \cos(t) \\ y(t) &= t\,e^t \end{cases}$$

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Given the following system of ODE:

$$\begin{cases} x''(t) + y(t) = t^2 \\ y''(t) - y(t) = 2 - t^2 \\ x(0) = 0, \qquad y(0) = 0, \qquad x'(0) = 1, \qquad y'(0) = A, \end{cases}$$

Compute constant A in such a way $x(1) = 1$.

Laplace transform:

$$\begin{cases} s^2 x(s) - 1 + y(s) &= \dfrac{2}{s^3} \\ s^2 y(s) - A - y(s) &= \dfrac{2}{(s-2)s^3} \end{cases}$$

Solution in s:

$$\begin{cases} x(s) &= \dfrac{-1 + s^2 - A}{s^2(s^2-1)}, \\ y(s) &= \dfrac{-2 + s^3 A + 2s^2}{s^3(s^2-1)}, \end{cases}$$

Solution in t:

$$\begin{cases} x(t) &= -A\sinh(t) + (1+A)t, \\ y(t) &= t^2 + A\sinh(t), \end{cases}$$

Constant A

$$A = 0$$

6

Compute matrix exponential of the following matrix:

$$\boldsymbol{A} = \begin{pmatrix} 9 & 5 & 4 \\ -12 & -7 & -8 \\ 4 & 3 & 5 \end{pmatrix}$$

Eigenvalues

$\lambda_1 = 1, \qquad \lambda_2 = 3, \qquad \lambda_3 = 3,$

Matrix exponential

$$e^{\boldsymbol{A}} = \begin{pmatrix} 94.96 & 54.83 & 34.74 \\ -149.8 & -89.53 & -69.49 \\ 57.55 & 37.46 & 37.46 \end{pmatrix}$$

7

Given the following function

$$f(x) = \begin{cases} 0 & \text{for } -\pi \le x < -\pi/2 \\ \cos x & \text{for } -\pi/2 \le x < \pi/2 \\ 0 & \text{for } \pi/2 \le x < \pi \end{cases}$$

defined for $x \in (-\pi, \pi)$ and extended periodically. Compute the coefficients of Fourier series:

$$a_0 = \frac{2}{\pi}$$

$$a_k = -2\frac{\cos(\pi k/2)}{(k^2-1)\pi} \begin{cases} \frac{2(-1)^(1+m)}{(4m^2-1)\pi} & \text{for } k = 2m \\ 0 & \text{for } k = 2m+1 \end{cases}$$

$$b_k = 0,$$