$$\begin{cases} x_{k+1} = x_k + y_k + 1 - k\,2^k \\ y_{k+1} = y_k + w_k + k\,2^k + 2^{k+1} - 1 \\ w_{k+1} = w_k + y_k + x_k - k\,2^k - k \end{cases} \qquad \begin{cases} x_0 = 0 \\ y_0 = 0 \\ w_0 = 1 \end{cases}$$

$$\begin{cases} (z-1)X - Y = \dfrac{z}{z-1} - \dfrac{2z}{(z-2)^2} & (\mathrm{I}) \Rightarrow Y = (z-1)X - \dfrac{z}{z-1} + \dfrac{2z}{(z-2)^2} \\ (z-1)Y - W = \dfrac{2z}{(z-2)^2} + \dfrac{2z}{z-2} - \dfrac{z}{z-1} & (\mathrm{II}) \Rightarrow W = (z-1)Y - \dfrac{2z}{(z-2)^2} - \dfrac{2z}{z-2} + \dfrac{z}{z-1} \\ (z-1)W - z - Y - X = -\dfrac{2z}{(z-2)^2} - \dfrac{z}{(z-1)^2} & (\mathrm{III}) \end{cases}$$

uso (I) in II $\Rightarrow$ $W = (z-1)^2 X - z + \dfrac{2z(z-1)}{(z-2)^2} - \dfrac{2z}{(z-2)^2} - \dfrac{2z}{z-2} + \dfrac{z}{z-1}$

Writing Y and W in function of X and collecting yields:

$$\left((z-1)^3 X - z(z-1) + \frac{2z(z-1)^2}{(z-2)^2} - \frac{2z(z-1)}{(z-2)^2} - \frac{2z(z-1)}{z-2} \cancel{+z} \right) \cancel{-z} - (z-1)X + \frac{z}{z-1} \cancel{- \frac{2z}{(z-2)^2}} - X \cancel{+ \frac{2z}{(z-2)^2}} + \frac{z}{(z-1)^2} = 0$$

Isolo X

$$X\left((z-1)^3 - (z-1) - 1 \right) = z(z-1) - \frac{2z(z-1)^2}{(z-2)^2} + \frac{2z(z-1)}{(z-2)^2} + \frac{2z(z-1)}{z-2} - \frac{z}{z-1} - \frac{z}{(z-1)^2}$$

$$X\left(z^3 - 3z^2 + 3z - 1 - z + 1 - 1 \right) = z(z-1) + \frac{2z(z-1)}{(z-2)^2}(1 - z + 1) + \frac{2z(z-1)}{z-2} - \frac{z}{z-1} - \frac{z}{(z-1)^2}$$

$$X\left(z^3 - 3z^2 + 2z - 1 \right) = \frac{1}{(z-1)^2(z-2)^2} \cdot \left[z(z-1)^3(z-2)^2 - 2z(z-1)^3(z-2) + 2z(z-1)^3(z-2) - z(z-1)(z-2)^2 - z(z-2)^2 \right]$$

$$X\left(z^3 - 3z^2 + 2z - 1 \right) = \frac{1}{(z-1)^2(z-2)^2} \cdot \left[z(z-1)^3(z-2)(z-2-2+2) - z(z-2)^2(z-1+1) \right]$$

$$= \frac{1}{(z-1)^2(z-2)^2} \cdot \left[z(z-1)^3(z-2)^2 - z^2(z-2)^2 \right] =$$

$$= \frac{z(z-2)^2}{(z-1)^2(z-2)^2}\left((z-1)^3 - z \right) = \frac{z(z-2)^2}{(z-1)^2(z-2)^2}\left(z^3 - 3z^2 + 3z - 1 - z \right) = X\left(z^3 - 3z^2 + 2z - 1 \right)$$

da cui $\boxed{X = \dfrac{z}{(z-1)^2}}$. Using X in (I) $\quad Y = \dfrac{z}{(z-1)^2}(z-1) - \dfrac{z}{z-1} + \dfrac{2z}{(z-2)^2} =$

$= \dfrac{z}{z-1} - \dfrac{z}{z-1} + \dfrac{2z}{(z-2)^2} \Rightarrow \boxed{Y = \dfrac{2z}{(z-2)^2}}$. the expression for W is

$$W = \frac{2z}{(z-2)^2}(z-1) - \frac{2z}{(z-2)^2} - \frac{2z}{z-2} + \frac{z}{z-1} = \frac{2z(z-1) - 2z}{(z-2)^2} - \frac{2z}{z-2} + \frac{z}{z-1} =$$

$$= \frac{2z(z-2)}{(z-2)^2} - \frac{2z}{z-2} + \frac{z}{z-1} = \frac{2z}{z-2} - \frac{2z}{z-2} + \frac{z}{z-1} \Rightarrow \boxed{W = \frac{z}{z-1}}$$