Numerical Methods for Dynamic System and Control del 27/08/2012

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Solve the following optimal control problem

$$\begin{cases} \text{Minimize:} \quad \displaystyle\int_0^1 u(t)^2 \,\mathrm{d}t \\ x'(t) = x(t) + u(t) \\ x(0) = 0, \quad x(1) = 0, \end{cases}$$

The Boundary Value Problem (BVP):

The optimal control $u(t)$:

The solution of the BVP:

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Study the following constrained minimization problem.

$$\text{minimize} \quad f(x,y) = x\,y, \qquad x \geq y^2 - 1, \qquad y^3 \geq x - 1,$$

KKT system of first order condition:

$$\begin{cases} y - \mu_1 + \mu_2 &= 0 \\ x + 2\mu_1 y - 3\mu_2 y^2 &= 0 \\ \mu_1(x - y^2 + 1) &= 0 \\ \mu_2(y^3 - x + 1) &= 0 \end{cases}$$

Solutions of KKT system:

$$\begin{cases} x = y = \mu_1 = \mu_2 = 0 \\ x = -\frac{2}{3}, \quad y = \mu_1 = \frac{\sqrt{3}}{3}, \quad \mu_2 = 0 \\ x = \frac{3}{4}, \quad y = -\frac{2^{1/3}}{2}, \quad \mu_1 = 0, \quad \mu_2 = \frac{2^{-1/3}}{2} \end{cases}$$

Discussion of the stationary point: $x = -\frac{2}{3}, \quad y = \mu_1 = \frac{\sqrt{3}}{3}, \quad \mu_2 = 0,$

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Solve the following recurrence

$$
\begin{aligned}
x_{k+1} &= x_k + y_k + 1 - k\,2^k, & x_0 &= 0 \\
y_{k+1} &= y_k + z_k + k\,2^k + 2^(k+1) - 1, & y_0 &= 0 \\
z_{k+1} &= z_k + y_k + x_k - k\,2^k - k, & z_0 &= 1
\end{aligned}
$$

$\mathcal{Z}-$transform:

$$
\begin{aligned}
(\zeta-1)x(\zeta) - y(\zeta) &= \frac{\zeta}{\zeta-1} - \frac{2\zeta}{(\zeta-2)^2} \\
(\zeta-1)y(\zeta) - z(\zeta) &= \frac{2\zeta/(\zeta-2)^2}{+}\frac{2\zeta}{\zeta-2} - \frac{\zeta}{\zeta-1} \\
(\zeta-1)z(\zeta) - \zeta - y(\zeta) - x(\zeta) &= -\frac{2\zeta}{(\zeta-2)^2} - \frac{\zeta}{(\zeta-1)^2}
\end{aligned}
$$

Solution in $\mathcal{Z}$:

$$
\begin{aligned}
x(\zeta) &= \frac{\zeta}{(\zeta-1)^2}, \\
y(\zeta) &= \frac{2\zeta}{(\zeta-2)^2}, \\
z(\zeta) &= \frac{\zeta}{\zeta-1},
\end{aligned}
$$

Solution in k

$$
\begin{aligned}
x_k &= k, \\
y_k &= k\,2^k, \\
z_k &= 1.
\end{aligned}
$$

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Given the following system of ordinary differential equations

$$\begin{aligned} y''(t) + x(t) &= 2 + t\cos(t) \\ x'(t) - y(t) &= \cos(t) - t\sin(t) - t^2 \\ x(0) = 0, \quad y(0) &= 0, \quad y'(0) = 0, \end{aligned}$$

Compute:

The Laplace transform:

$$\begin{cases} s^2 y(s) + x(s) &= \dfrac{2}{s} + \dfrac{s^2-1}{(s^2+1)^2} \\ s\,x(s) - y(s) &= \dfrac{s^6 - 3s^4 - 4s^2 - 2}{(s^2+1)^2 s^3} \end{cases}$$

Solution in s:

$$\begin{cases} x(s) &= \dfrac{s^2-1}{s^4+2s^2+1} = \dfrac{s^2-1}{(s^2+1)^2} \\ y(s) &= \dfrac{2}{s^3} \end{cases}$$

Solution in t:

$$\begin{cases} x(t) &= t^2 \\ y(t) &= t\cos(t) \end{cases}$$

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Given the following system of ODE:

$$\begin{cases} x''(t) + x'(t) + y(t) = 1 + \cos(t) \\ y'(t) - y(t) = -\sin(t) - \cos(t) \\ x(0) = 0, \qquad y(0) = 1, \qquad x'(0) = A, \end{cases}$$

Compute constant A in such a way $x(1) = 1$.

Laplace transform:

$$\begin{cases} s^2 x(s) - A + s\,x(s) + y(s) &= \dfrac{2s^2+1}{s(s^2+1)} \\ s\,y(s) - 1 - y(s) &= -\dfrac{1+s}{s^2+1} \end{cases}$$

Solution in s:

$$\begin{cases} x(s) &= \dfrac{sA+1}{s^2(1+s)}, \\ y(s) &= \dfrac{s}{s^2+1}, \end{cases}$$

Solution in t:

$$\begin{cases} x(t) &= -1 + A + t + (1-A)e^{-t} \\ y(t) &= \cos(t) \end{cases}$$

Constant A

$$A = 1$$

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Compute matrix exponential of the following matrix:

$$\boldsymbol{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 3 & 2 & 3 \end{pmatrix}$$

Eigenvalues

$\lambda_1 = 2, \qquad \lambda_2 = 1, \qquad \lambda_3 = 1,$

Matrix exponential

$$e^{\boldsymbol{A}} = \begin{pmatrix} 2.718 & 0 & 0 \\ -5.86 & -1.953 & -4.671 \\ 19.87 & 9.344 & 12.06 \end{pmatrix}$$

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Given the following function

$$f(x) = \begin{cases} 0 & \text{for } -\pi \le x < -\pi/2 \\ \pi^2/4 - x^2 & \text{for } -\pi/2 \le x < \pi/2 \\ 0 & \text{for } \pi/2 \le x < \pi \end{cases}$$

defined for $x \in (-\pi, \pi)$ and extended periodically. Compute the coefficients of Fourier series:

$$a_0 = \frac{\pi^2}{6}$$

$$a_k = 2\frac{k\cos(k\pi/2)\pi - 2\sin(k\pi/2)}{k^3\pi} = \begin{cases} \dfrac{(-1)^m}{2m^2} & \text{for } k = 2m \\ \dfrac{4(-1)^m}{(2m+1)^3\pi} & \text{for } k = 2m+1 \end{cases}$$

$b_k = 0,$