Exercitation 1

Numerical Methods for Dynamical Systems and Control

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1 Exercise 1

Resolve the following system of ODE for $t > 0$.

$$\begin{cases} x'(t) & + & x''(t) & = & \sin(t) \\ y'(t) & - & x''(t) & = & \cos(t) \\ x(0) = 1 & & x'(0) = 0 & & y(0) = -1 \end{cases} \tag{1}$$

1.1 Solution with ODE techniques

Starting from the first equation of (1), the solution can be written as the sum of the solution of the homogeneous equation x_h and the solution of the particular equation x_p. One calculates the polynomial associated to the homogeneous equation ($x_h := x'' + x' = 0$) $\lambda^2 + \lambda = 0$: it has roots $\lambda_1 = 0$, $\lambda_2 = -1$. Thus the solution of the homogeneous equation x_h is $c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$ for two constants c_1 and c_2 to be determined from the initial conditions (ICS). The particular solution x_p is a combination of sines and cosines, $x_p := \alpha\cos(t) + \beta\sin(t)$ for constants $\alpha,\ \beta$. These expressions can be summarized in

$$\begin{array}{rclcrcl} x_h & = & c_1 + c_2 e^{-t} & \qquad & x_p & = & \alpha\cos(t) + \beta\sin(t) \\ x_h' & = & -c_2 e^{-t} & & x_p' & = & -\alpha\sin(t) + \beta\cos(t) \\ x_h'' & = & c_2 e^{-t} & & x_p'' & = & -\alpha\cos(t) - \beta\sin(t) \end{array} \tag{2}$$

and $x(t) = x_h(t) + x_p(t)$. The unknown constants are derived as follows. The first equation of (1) leads to

$$x'' + x' = c_2 e^{-t} - \alpha\cos(t) - \beta\sin(t) - c_2 e^{-t} - \alpha\sin(t) + \beta\cos(t) = \sin(t), \tag{3}$$

solving for $\alpha,\ \beta$ one has

$$\begin{cases} -\alpha - \beta & = & 1 \\ \beta - \alpha & = & 0 \end{cases} \qquad \Rightarrow \alpha = \beta = -\frac{1}{2}. \tag{4}$$

Constants $c_1,\ c_2$ are obtained from the ICS:

$$\begin{array}{rcccl} 1 = x(0) & = & \left(c_1 + c_2 e^{-t} - \frac{1}{2}\cos(t) - \frac{1}{2}\sin(t)\right)\Big|_{t=0} & = & c_1 + c_2 - \frac{1}{2} \\ 0 = x'(0) & = & \left(-c_2 e^{-t} - \frac{1}{2}\cos(t) + \frac{1}{2}\sin(t)\right)\Big|_{t=0} & = & -c_2 - \frac{1}{2} \end{array} \tag{5}$$

thus $c_1 = 2$ and $c_2 = -1/2$. So the solution of the first equation is

$$\boxed{x(t) = 2 - \frac{e^{-t} + \sin(t) + \cos(t)}{2}} \tag{6}$$

The second equation of (1) is easier to be solved.

$$y' - x'' = \cos(t) \Rightarrow y'(t) = -\frac{1}{2}e^{-t} + \frac{3}{2}\cos(t) + \frac{1}{2}\sin(t) \tag{7}$$

and integrating both sides

$$y(t) = \frac{1}{2}e^{-t} + \frac{3}{2}\sin(t) - \frac{1}{2}\cos(t) + c \tag{8}$$

for a constant c that can be obtained by ICS, i.e. $c = -1$. So the second equation has solution

$$\boxed{y(t) = -1 + \frac{e^{-t} + 3\sin(t) + \cos(t)}{2}} \tag{9}$$

1.2 Solution with Laplace Transform

Applying the Laplace tranform to both sides of (1) gives

$$\left\{ \begin{array}{rcl} sX(s) - x(0) + s^2X(s) - sx(0) - x'(0) & = & \frac{1}{s^2+1} \\ \\ sY(s) - y(0) - s^2X(s) + sx(0) + x'(0) & = & \frac{s}{s^2+1} \end{array} \right. \tag{10}$$

substituting the ICS leads to

$$\left\{ \begin{array}{rcl} sX(s) - 1 + s^2X(s) - s & = & \frac{1}{s^2+1} \\ \\ sY(s) + 1 - s^2X(s) + s & = & \frac{s}{s^2+1}. \end{array} \right. \tag{11}$$

Simplifying and solving for $X(s),\ Y(s)$ one obtains

$$\left\{ \begin{array}{rcl} X(s) & = & \frac{(s^2+1)(s+1)+1}{s(s^2+1)(s+1)} \\ \\ Y(s) & = & -\frac{1}{s} + sX(s) - 1 + \frac{1}{s^2+1} \end{array} \right. \tag{12}$$

The next step is to perform partial fraction decomposition of the right hand side of (12), for the first equation for now.

$$X(s) = \frac{2}{s} - \frac{1}{2}\frac{1}{s+1} + \left(-\frac{1}{4} + \frac{1}{4}\boldsymbol{i}\right)\frac{1}{s+\boldsymbol{i}} + \left(-\frac{1}{4} - \frac{1}{4}\boldsymbol{i}\right)\frac{1}{s-\boldsymbol{i}} \tag{13}$$

Remark 1. *One can observe that the coefficient for the partial fraction decomposition with respect to s^2+1 are conjugated. So it is sufficient to compute one coefficient and not both. Another decomposition could have been*

$$X(s) = \frac{2}{s} - \frac{1}{2}\frac{1}{s+1} + \frac{s+1}{2(s^2+1)}. \tag{14}$$

Now it is time to invert the Laplace transform in order to write the solution in the time domain. From the tables of transformed functions one finds out that

$$\mathcal{L}^{-1}\{X(s)\}(t) = 2 - \frac{1}{2}e^{-t} + \left(-\frac{1}{4} + \frac{1}{4}\boldsymbol{i}\right)e^{\boldsymbol{i}t} + \left(-\frac{1}{4} - \frac{1}{4}\boldsymbol{i}\right)e^{-\boldsymbol{i}t} \tag{15}$$

Remark 2. *The Euler relation states that*

$$e^{\boldsymbol{i}x} = \cos(x) + \boldsymbol{i}\sin(x)$$

so the expression $\left(-\frac{1}{4} + \frac{1}{4}\boldsymbol{i}\right)e^{\boldsymbol{i}t} + \left(-\frac{1}{4} - \frac{1}{4}\boldsymbol{i}\right)e^{-\boldsymbol{i}t}$ *becomes*

$$-\frac{1}{2}\cos(t) - \frac{1}{2}\sin(t).$$

So, the first differential equation has solution

$$\boxed{x(t) = 2 - \frac{e^{-t} + \sin(t) + \cos(t)}{2}} \tag{16}$$

After substituting the solution in the frequency domain (13) in the second equation of (12) the result is

$$\begin{aligned}
Y(s) &= -\frac{1}{s} + \frac{s^3+s^2+s+2}{(s^2+1)(s+1)} - 1 + \frac{1}{s^2+1} \\
&= -\frac{1}{s} + 1 + \frac{1}{(s^2+1)(s+1)} - 1 + \frac{1}{s^2+1} \\
&= -\frac{1}{s} + \frac{1}{2}\frac{1}{s+1} - \frac{s-1}{2(s^2+1)} + \frac{1}{s^2+1} \\
&= -\frac{1}{s} + \frac{1}{2}\frac{1}{s+1} + \frac{3}{2}\frac{1}{s^2+1} - \frac{1}{2}\frac{s}{s^2+1}.
\end{aligned} \tag{17}$$

Taking the inverse Laplace transform one has the solution in the time domain,

$$\boxed{y(t) = -1 + \frac{e^{-t} + 3\sin(t) - \cos(t)}{2}.} \tag{18}$$

□

2 Exercise 2

Resolve the following ODE for $t > 0$.

$$\begin{cases} y'''(t) & = & -1 \\ y(0) = 1 & y'(0) = 0 & y''(0) = A \end{cases} \tag{19}$$

such that $y(1) = 1$.

2.1 Solution with Laplace Transform

The first thing to do is to transform the equation from the time domain in the frequency domain via the Laplace transform.

$$\mathcal{L}\{y'''\}(t) = s^3 Y(s) - s^2 y(0) - s y'(0) - y''(0) = -\frac{1}{s}. \tag{20}$$

The substitution of the ICS yields

$$s^3 Y(s) - s^2 - A = -\frac{1}{s} \qquad \Rightarrow \qquad Y(s) = \frac{A + s^2 - 1/s}{s^3}. \tag{21}$$

The partial fraction decomposition is easy to write down,

$$Y(s) = \frac{A}{s^3} + \frac{1}{s} - \frac{1}{s^4} \tag{22}$$

and also the inversion is straight forward:

$$y(t) = \mathcal{L}\{Y(s)\}(t) = \frac{1}{2} A t^2 + 1 - \frac{1}{6} t^3. \tag{23}$$

A can be evaluated imposing the condition $y(1) = 1$

$$y(1) = \frac{1}{2} A + 1 - \frac{1}{6} \qquad \Rightarrow \qquad A = \frac{1}{3} \tag{24}$$

So, the desired solution is

$$\boxed{y(t) = \frac{1}{6} t^2 + 1 - \frac{1}{6} t^3 = \frac{1}{6} t^2 (1 - t) + 1.} \tag{25}$$

□

3 Exercise 3

Resolve the following ODE for $t > 0$.

$$\begin{cases} x''(t) - 3x'(t) + 2x(t) = \cos(t) - 3e^{5t} + t^2 + 1 \\ x'(0) = 1 \\ x''(0) = 2 \end{cases} \tag{26}$$

3.1 Solution with ODE techniques

The solution $x(t)$ is given by the sum of the homogeneous equation and the particular one, $x(t) = x_h(t) + x_p(t)$. The polynomial associated to the homogeneous equation is $\lambda^2 - 3\lambda + 2$ and its roots are $\{1, 2\}$, thus

$$x_h(t) = c_1 e^t + c_2 e^{2t} \qquad x_p(t) = \alpha\cos(t) + \beta\sin(t) + \gamma e^{5t} + \delta t^2 + \varepsilon t + \zeta g \tag{27}$$

for constants $c_1,\ c_2,\ \alpha,\ \beta,\ \gamma,\ \delta,\ \varepsilon,\ \zeta$. The derivatives of $x_p(t)$ are respectively

$$\begin{aligned} x_p'(t) &= -\alpha\sin(t) + \beta\cos(t) + 5\gamma e^{5t} + 2\delta t + \varepsilon \\ x_p''(t) &= -\alpha\cos(t) - \beta\sin(t) + 25\gamma e^{5t} + 2\delta \end{aligned} \tag{28}$$

The use of those expression in $x'' - 3x' + 2x = \cos(t) - 3e^{5t} + t^2 + 1$ leads to

$$\begin{aligned} &(-\alpha - 3\beta + 2\alpha - 1)\cos(t) + \\ &(-\beta + 3\alpha + 2\beta)\sin(t) + \\ &(25\gamma - 15\gamma + 2\gamma + 3)e^{5t} + \\ &(2\delta - 3\varepsilon + 2\zeta - 1)t^2 + \\ &(-6\delta + 2\varepsilon)t + \\ &(2\delta - 1) = 0 \end{aligned} \tag{29}$$

This simple linear system has the following solution:

$$\alpha = 1/10,\ \beta = -3/10,\ \gamma = -1/4,\ \delta = 1/2,\ \varepsilon = 3/2,\ \zeta = 9/4.$$

Plugging these values in (27) it is possible to compute the two constants $c_1,\ c_2$ from the ICS.

$$1 = x'(0) \Rightarrow c_1 + 2c_2 = \frac{21}{20} \qquad 2 = x''(0) \Rightarrow c_1 + 4c_2 = \frac{147}{20} \tag{30}$$

The two solutions are $c_1 = -\frac{21}{4},\ c_2 = \frac{63}{20}$ thus the solution of the ODE is

$$\boxed{x(t) = -\frac{21}{4}e^t + \frac{63}{20}e^{2t} + \frac{1}{10}\cos(t) - \frac{3}{10}\sin(t) - \frac{1}{4}e^{5t} + \frac{1}{2}t^2 + \frac{3}{2}t + \frac{9}{4}} \tag{31}$$

3.2 Solution with Laplace Transform

The Laplace transform of (26) is

$$s^2X(s) - sx(0) - 1 - 3sX(s) + 3x(0) + 2X(s) = \frac{s}{s^2+1} - \frac{3}{s-5} + \frac{2}{s^3} + \frac{1}{s} \tag{32}$$

collecting $X(s)$ at the LHS

$$\begin{aligned} X(s) \quad = \quad & \frac{x(0)(s-3)}{(s-1)(s-2)} + \frac{1}{(s-1)(s-2)} + \frac{s}{(s^2+1)(s-1)(s-2)} \\ & -\frac{3}{(s-1)(s-2)(s-5)} + \frac{2}{s^3(s-1)(s-2)} + \frac{1}{s(s-1)(s-2)} \end{aligned} \tag{33}$$

The decomposition in partial fractions of the six terms of the sum is

- $\dfrac{x(0)(s-3)}{(s-1)(s-2)} = \dfrac{2x(0)}{s-1} - \dfrac{x(0)}{s-2}$
- $\dfrac{1}{(s-1)(s-2)} = \dfrac{-1}{s-1} + \dfrac{1}{s-2}$
- $\dfrac{s}{(s^2+1)(s-1)(s-2)} = \dfrac{\frac{1}{20} + \frac{3}{20}\boldsymbol{i}}{s-\boldsymbol{i}} + \dfrac{\frac{1}{20} - \frac{3}{20}\boldsymbol{i}}{s+\boldsymbol{i}} - \dfrac{1}{2}\dfrac{1}{s-1} + \dfrac{2}{5}\dfrac{1}{s-2}$
  In fact for example the coefficient of $s+\boldsymbol{i}$ is

  $$\frac{-\boldsymbol{i}}{-2\boldsymbol{i}(-\boldsymbol{i}-1)(-\boldsymbol{i}-2)} = \frac{1}{2(1+3\boldsymbol{i})}\frac{1-3\boldsymbol{i}}{1-3\boldsymbol{i}} = \frac{1}{20} - \frac{3}{20}\boldsymbol{i}.$$

  The coefficient of $s-\boldsymbol{i}$ is its conjugate.
- $-\dfrac{3}{(s-1)(s-2)(s-5)} = -3\left(\dfrac{\frac{1}{4}}{s-1} - \dfrac{\frac{1}{3}}{s-2} + \dfrac{\frac{1}{12}}{s-5}\right)$
- $\dfrac{2}{s^3(s-1)(s-2)} = \dfrac{1}{s^3} + \dfrac{\frac{3}{2}}{s^2} + \dfrac{\frac{7}{4}}{s} - \dfrac{2}{s-1} + \dfrac{\frac{1}{4}}{s-2}$
  The coefficients of $s^3,\ s-1,\ s-2$ are easy, the main problem in this partial fraction decomposition are the powers $s^2,\ s$. The value of the pole ($s=0$) can not be substituted as for the coefficient of s^3 because of the singularity. The problem can be resolved passing through a 2×2 linear system. It is enough to evaluate the whole term in two different points, say 3 and 4 obtaining two linear equations:

  $$\frac{2}{3^3(3-1)(3-2)} = \frac{1}{3^3} + \frac{A}{3^2} + \frac{B}{3} - \frac{2}{3-1} + \frac{\frac{1}{4}}{3-2}$$

  $$\frac{2}{4^3(4-1)(4-2)} = \frac{1}{4^3} + \frac{A}{4^2} + \frac{B}{4} - \frac{2}{4-1} + \frac{\frac{1}{4}}{4-2}$$

The simplified system is

$$-\frac{3}{4}+\frac{1}{9}A+\frac{1}{3}B=0 \qquad -\frac{17}{32}+\frac{1}{16}A+\frac{1}{4}B=0$$

and has solution $A=3/2$ and $B=7/4$.

- $\dfrac{1}{s(s-1)(s-2)}=\dfrac{\frac{1}{2}}{s}-\dfrac{1}{s-1}+\dfrac{\frac{1}{2}}{s-2}$

The partial fraction decomposition can be written thus as

$$X(s)=\frac{\left(2x(0)-\frac{21}{4}\right)}{s-1}+\frac{\left(\frac{63}{20}-x(0)\right)}{s-2}-\frac{\frac{1}{4}}{s-5}+\frac{\frac{1}{20}+\frac{3}{20}\boldsymbol{i}}{s-\boldsymbol{i}}+\frac{\frac{1}{20}-\frac{3}{20}\boldsymbol{i}}{s+\boldsymbol{i}}+\frac{1}{s^3}+\frac{\frac{3}{2}}{s^2}+\frac{\frac{9}{4}}{s} \tag{34}$$

Taking the inverse Laplace transform, one has

$$x(t)=\left(2x(0)-\frac{21}{4}\right)e^{t}+\left(\frac{63}{20}-x(0)\right)e^{2t}-\frac{1}{4}e^{5t}+\frac{1}{10}\cos(t)-\frac{3}{10}\sin(t)+\frac{1}{2}t^2+\frac{3}{2}t+\frac{9}{4} \tag{35}$$

If one differentiates the previous equation twice, will find the value of $x(0)=0$. Putting $x(0)=0$ transforms the (35) in the (31).

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