Exercitation 2

Numerical Methods for Dynamical Systems and Control

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1 Exercise 1

Find $\mathcal{L}^{-1}\left\{\dfrac{5s^2-15s-11}{(s+1)(s-2)^3}\right\}$.

1.1 Solution

The first thing to do is the partial fraction decomposition

$$\frac{5s^2-15s-11}{(s+1)(s-2)^3} = \frac{A}{s+1} + \frac{B}{(s-2)^3} + \frac{C}{(s-2)^2} + \frac{D}{(s-2)}. \tag{1}$$

Multiplying both sides by $s+1$ and putting $s=-1$ then $A=-\frac{1}{3}$. Multiply both sides by $(s-2)^3$ and put $s=2$ to get $B=-7$. This method fails to determine $C,\ D$. However since A and B are known, one has

$$\frac{5s^2-15s-11}{(s+1)(s-2)^3} = \frac{-\frac{1}{3}}{s+1} + \frac{-7}{(s-2)^3} + \frac{C}{(s-2)^2} + \frac{D}{(s-2)}. \tag{2}$$

To determine $C,\ D$ one can substitute two values for s, say $s=0$ and $s=1$ from which can find respectively

$$\frac{11}{8} = -\frac{1}{3} + \frac{7}{8} + \frac{C}{4} - \frac{D}{2} \qquad\qquad \frac{21}{2} = -\frac{1}{6} + 7 + C - D \tag{3}$$

i.e. $3C-6D=10$ and $3C-3D=11$, from which $C=4$, $D=\frac{1}{3}$, thus

$$\mathcal{L}^{-1}\left\{\frac{5s^2-15s-11}{(s+1)(s-2)^3}\right\} = \mathcal{L}^{-1}\left\{\frac{-\frac{1}{3}}{s+1} + \frac{-7}{(s-2)^3} + \frac{4}{(s-2)^2} + \frac{\frac{1}{3}}{(s-2)}\right\}. \tag{4}$$

So the solution is

$$\boxed{-\frac{1}{3}e^{-t} - \frac{7}{2}t^2e^{2t} + 4te^{2t} + \frac{1}{3}e^{2t}.} \tag{5}$$

Remark 1. *Let see another method for computing $C,\ D$. Multiplying both sides of (2) by s and letting $s\to\infty$ one finds $0=-\frac{1}{3}+D$ which gives $D=\frac{1}{3}$. Then C can be found as above letting $s=0$. This method can be used when there are some repeated linear factors.*

□

2 Exercise 2

Find $\mathcal{L}^{-1}\left\{\dfrac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}\right\}$.

2.1 Solution - method 1

The first thing to do is the partial fraction decomposition

$$\frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)} = \frac{As+B}{s^2+2s+2} + \frac{Cs+D}{s^2+2s+5} \tag{6}$$

Multiplying both sides by $(s^2+2s+2)(s^2+2s+5)$ one has

$$\begin{aligned} s^2+2s+3 &= (As+B)(s^2+2s+5)+(Cs+D)(s^2+2s+2) \\ &= (A+C)s^3+(2A+B+2C+D)s^2+(5A+2B+2C+2D)s+5B+2D \end{aligned} \tag{7}$$

This leads to the following linear system

$$\begin{cases} A+C=0 \\ 2A+B+2C+D=1 \\ 5A+2B+2C+2D=2 \\ 5B+2D=3 \end{cases} \tag{8}$$

Solving, $A=0,\ B=\dfrac{1}{3},\ C=0,\ D=\dfrac{2}{3}$, thus

$$\begin{aligned} \mathcal{L}^{-1}\left\{\frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}\right\} &= \mathcal{L}^{-1}\left\{\frac{\frac{1}{3}}{s^2+2s+2}+\frac{\frac{2}{3}}{s^2+2s+5}\right\} \\ &= \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+1}\right\}+\frac{2}{3}\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+4}\right\} \\ &= \frac{1}{3}e^{-t}\sin(t)+\frac{2}{3}\frac{1}{2}e^{-t}\sin(2t) \end{aligned} \tag{9}$$

In facts for $t>0$ and $\mathrm{Re}(s)>-\alpha$

$$\mathcal{L}\left\{e^{-\alpha t}\sin(\omega t)\right\} = \frac{\omega}{(s+\alpha)^2+\omega^2} = \frac{1}{s^2+As+B} \tag{10}$$

where $\alpha=\frac{A}{2}$ and $\omega=\sqrt{B-\frac{A^2}{4}}$. So, the solution is

$$\boxed{\frac{1}{3}e^{-t}(\sin(t)+\sin(2t)).} \tag{11}$$

2.2 Solution - method 2

Let $s = 0$ and use the initial value theorem (multiply by s and let $s \to \infty$) in (6), then respectively

$$\frac{3}{10} = \frac{B}{2} + \frac{D}{5} \qquad\qquad 0 = A + C \tag{12}$$

Let $s = 1$ and $s = -1$ in (6), then respectively

$$\frac{3}{20} = \frac{A+B}{5} + \frac{C+D}{8} \qquad\qquad \frac{1}{2} = -A + B + \frac{D-C}{4}. \tag{13}$$

These four equations lead to a linear system which gives $A = 0,\ B = \frac{1}{3},\ C = 0,\ D = \frac{2}{3}$ as in method 1. This illustrates the case of non-repeated quadratic factors.

2.3 Solution - method 3

Since the roots of $s^2 + 2s + 2 = 0$ are $-1 \pm \boldsymbol{i}$ and similarly the roots of $s^2 + 2s + 5 = 0$ are $-1 \pm 2\boldsymbol{i}$ one can write

$$\frac{s^2 + 2s + 3}{(s^2 + 2s + 2)(s^2 + 2s + 5)} = \frac{s^2 + 2s + 3}{(s + 1 - \boldsymbol{i})(s + 1 + \boldsymbol{i})(s + 1 - 2\boldsymbol{i})(s + 1 + 2\boldsymbol{i})} \tag{14}$$

which leads to a linear partial fraction decomposition

$$\frac{A}{(s + 1 - \boldsymbol{i})} + \frac{B}{(s + 1 + \boldsymbol{i})} + \frac{C}{(s + 1 - 2\boldsymbol{i})} + \frac{D}{(s + 1 + 2\boldsymbol{i})}. \tag{15}$$

Substituting the first root yields

$$\frac{(-1 + \boldsymbol{i})^2 + 2(-1 + \boldsymbol{i}) + 3}{(-1 + \boldsymbol{i} + 1 + \boldsymbol{i})(-1 + \boldsymbol{i} + 1 - 2\boldsymbol{i})(-1 + \boldsymbol{i} + 1 + 2\boldsymbol{i})} = \frac{1 - 2\boldsymbol{i} - 1 - 2 + 2\boldsymbol{i} + 3}{2\boldsymbol{i}(-\boldsymbol{i})(3\boldsymbol{i})} = \frac{1}{6\boldsymbol{i}} \tag{16}$$

therefore $A = \frac{1}{6i}$ and $B = -\frac{1}{6i}$.
Substituting the root $-1 + 2\boldsymbol{i}$ yields

$$\frac{(-1 + 2\boldsymbol{i})^2 + 2(-1 + 2\boldsymbol{i}) + 3}{(-1 + 2\boldsymbol{i} + 1 - \boldsymbol{i})(-1 + 2\boldsymbol{i} + 1 + \boldsymbol{i})(-1 + 2\boldsymbol{i} + 1 + 2\boldsymbol{i})} = \frac{1 - 4\boldsymbol{i} - 4 - 2 + 4\boldsymbol{i} + 3}{\boldsymbol{i}(3\boldsymbol{i})(4\boldsymbol{i})} = \frac{1}{6\boldsymbol{i}} \tag{17}$$

therefore $C = \frac{1}{6i}$ and $D = -\frac{1}{6i}$. The inverse Laplace transform is

$$\begin{aligned} \frac{e^{-(1-\boldsymbol{i})t}}{6\boldsymbol{i}} - \frac{e^{-(1+\boldsymbol{i})t}}{6\boldsymbol{i}} + \frac{e^{-(1-2\boldsymbol{i})t}}{6\boldsymbol{i}} - \frac{e^{-(1+2\boldsymbol{i})t}}{6\boldsymbol{i}} &= \frac{1}{3}e^{-t}\left(\frac{e^{\boldsymbol{i}t} - e^{-\boldsymbol{i}t}}{2\boldsymbol{i}}\right) + \frac{1}{3}e^{-t}\left(\frac{e^{2\boldsymbol{i}t} - e^{-2\boldsymbol{i}t}}{2\boldsymbol{i}}\right) \\ &= \frac{1}{3}e^{-t}\sin(t) + \frac{1}{3}e^{-t}\sin(2t) \\ &= \frac{1}{3}e^{-t}(\sin(t) + \sin(2t)) \end{aligned} \tag{18}$$

This shows that the case of non-repeated quadratic factors can be reduced to non-repeated linear factors using complex numbers.

□

3 Exercise 3

Prove that if $\mathcal{L}\{f(t)\} = F(s)$, then

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(u)\ du$$

3.1 Proof

Let $g(t) = \frac{f(t)}{t}$, then $f(t) = t\,g(t)$. Taking the Laplace transform of both sides one has

$$\mathcal{L}\left\{f(t)\right\} = -\frac{d}{ds}\mathcal{L}\left\{g(t)\right\} \qquad \text{or} \qquad F(s) = -\frac{dG(s)}{ds}. \tag{19}$$

Then integrating

$$G(s) = -\int_{-\infty}^s F(u)\ du = \int_s^\infty F(u)\ du \tag{20}$$

and this is the same of

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(u)\ du. \tag{21}$$

□

4 Exercise 4

Show that

$$\int_0^\infty \frac{\sin(t)}{t} = \frac{\pi}{2}.$$

4.1 Proof

Let $f(t) = \sin(t)$ so that $F(s) = \frac{1}{s^2+1}$. Remind that $\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(u)\ du$, then

$$\int_0^\infty \frac{\sin(t)}{t} = \lim_{s\to 0^+}\int_s^\infty \frac{du}{u^2+1} = \int_0^\infty \frac{du}{u^2+1} = \arctan(u)\Big|_o^\infty = \frac{\pi}{2}. \tag{22}$$

□

5 Exercise 5

Show that

$$\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}.$$

5.1 Proof

Consider $g(t) = \int_0^\infty e^{-tx^2}\,dx$, then taking the Laplace trandform of $g(t)$

$$\begin{aligned}
\mathcal{L}\{g(t)\} &= \int_0^\infty e^{-st}\int_0^\infty e^{-tx^2}\,dx\,dt \\
&= \int_0^\infty\int_0^\infty e^{-st}e^{-tx^2}\,dt\,dx \\
&= \int_0^\infty\int_0^\infty e^{-(s+x^2)t}\,dt\,dx \\
&= \int_0^\infty \mathcal{L}\left\{e^{-(s+x^2)t}\right\}\,dx \\
&= \int_0^\infty \frac{1}{s+x^2}\,dx.
\end{aligned} \tag{23}$$

Making the change of variable $\lambda = \frac{x}{\sqrt{s}}$ that implies $d\lambda = \frac{dx}{\sqrt{s}}$ the integral becomes

$$\int_0^\infty \frac{1}{s+x^2}\,dx = \frac{1}{\sqrt{s}}\int_0^\infty \frac{1}{1+\lambda^2}\,d\lambda = \frac{1}{\sqrt{s}}\arctan(\lambda)\Big|_0^\infty = \frac{\pi}{2\sqrt{s}} \tag{24}$$

Thus by inverting

$$\mathcal{L}^{-1}\left\{\frac{1}{\sqrt{s}}\right\} = \mathcal{L}^{-1}\left\{s^{-1/2}\right\} = \frac{t^{-1/2}}{\Gamma(1/2)} = \frac{t^{-1/2}}{\sqrt{\pi}}. \tag{25}$$

So finally

$$g(t) = \int_0^\infty e^{-tx^2}\,dx = \frac{\pi t^{-1/2}}{2\sqrt{\pi}} = \frac{1}{2}\sqrt{\pi}t^{-1/2} \tag{26}$$

and substituting $t = 1$

$$\boxed{\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}.} \tag{27}$$

□

6 Exercise 6

Resolve the Cauchy problem

$$\begin{cases} \boldsymbol{x}'(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{f}(t) \\ \boldsymbol{x}(0) = \boldsymbol{x_0} \end{cases} \tag{28}$$

where $\boldsymbol{x}, \boldsymbol{x_0}, \boldsymbol{f} \in \mathbb{R}^n$ and $\boldsymbol{A} \in \mathbb{R}^{n\times n}$.

6.1 Solution with Laplace Transform

Performing the Laplace transform yields

$$s\boldsymbol{X(s)} - \boldsymbol{x_0} = \boldsymbol{A}\boldsymbol{X(s)} + \boldsymbol{F}(s). \tag{29}$$

Solving for $\boldsymbol{X(s)}$ gives

$$(\boldsymbol{I}s - \boldsymbol{A})\boldsymbol{X(s)} = \boldsymbol{x_0} + \boldsymbol{F}(s), \tag{30}$$

where $\boldsymbol{I} \in \mathbb{R}^{n\times n}$, thus

$$\begin{aligned} \boldsymbol{X(s)} &= (\boldsymbol{I}s - \boldsymbol{A})^{-1}\boldsymbol{x_0} + (\boldsymbol{I}s - \boldsymbol{A})^{-1}\boldsymbol{F}(s) \\ &= \mathcal{L}\{e^{\boldsymbol{A}t}\}\boldsymbol{x_0} + \mathcal{L}\{e^{\boldsymbol{A}t}\}\mathcal{L}\{\boldsymbol{f}\}. \end{aligned} \tag{31}$$

Applying the inversion of the transform one has

$$\boxed{\boldsymbol{x}(t) = e^{\boldsymbol{A}t}\boldsymbol{x_0} + \textstyle\int_0^t e^{\boldsymbol{A}(t-s)}\boldsymbol{f}(s)\, ds.} \tag{32}$$

If $\boldsymbol{A}(t)$ is not a constant matrix, the homogeneous system is $\boldsymbol{x}' = \boldsymbol{A}(t)\boldsymbol{x}(t)$. To solve it one needs a primitive of $\boldsymbol{A}(t)$ which is $\int_0^t \boldsymbol{A}(s)\, ds$. Then $x_h(t) = e^{\int_0^t \boldsymbol{A}(s)\, ds}\boldsymbol{x_0}$. Adding the particular solution one has

$$x(t) = e^{\int_0^t \boldsymbol{A}(s)\, ds}\boldsymbol{x_0} + e^{\int_0^t \boldsymbol{A}(s)\, ds}\int_0^t \boldsymbol{f}(\xi)e^{-\int_0^t \boldsymbol{A}(s)\, ds}\, d\xi. \tag{33}$$

One should take care because in general matrices are non commutative and integrals can not be swapped.

6.2 Solution with ODE techniques

This is the standard method for a first order differential equation. One observes that multiplying by $e^{-\boldsymbol{A}t}$ both sides of (28) gives

$$e^{-\boldsymbol{A}t}\boldsymbol{x'}(t) = e^{-\boldsymbol{A}t}\boldsymbol{A}\boldsymbol{x}(t) + e^{-\boldsymbol{A}t}\boldsymbol{f}(t) \tag{34}$$

and from $\frac{d}{dt}\left(e^{-\boldsymbol{A}t}\boldsymbol{x}(t)\right) = e^{-\boldsymbol{A}t}\boldsymbol{f}(t)$ there is the classic formula

$$e^{-\boldsymbol{A}t}\boldsymbol{x}(t) = \boldsymbol{x_0} + \int_0^t e^{-\boldsymbol{A}(\xi)}\boldsymbol{f}(\xi)\, d\xi \tag{35}$$

which is exactly formula (32) obtained via the Laplace transform.

□

7 Review of partial fraction decomposition

Review of methods for obtaining partial fraction decomposition. There are four cases.

- A single real root,

$$\frac{P(s)}{(s-a)Q(s)} = \frac{A}{s-a} + \frac{q(s)}{Q(s)} \tag{36}$$

- Two complex roots (conjugated) ($\Delta = a^2 - 4b < 0$),

$$\frac{P(s)}{(s^2+as+b)Q(s)} = \frac{As+B}{s^2+as+b} + \frac{q(s)}{Q(s)} \tag{37}$$

- Repeated real roots $r > 1$,

$$\frac{P(s)}{(s-a)^r Q(s)} = \frac{A_r}{(s-a)^r} + \frac{A_{r-1}}{(s-a)^{r-1}} + \cdots + \frac{A_1}{s-a} + \frac{q(s)}{Q(s)} \tag{38}$$

- Repeated complex roots ($r > 1$ and $\Delta = a^2 - 4b < 0$),

$$\frac{P(s)}{(s^2+as+b)^r Q(s)} = \frac{A_r s + B_r}{(s^2+as+b)^r} + \frac{A_{r-1}s + B_{r-1}}{(s^2+as+b)^{r-1}} + \cdots + \frac{A_1 s + B_1}{s^2+as+b} + \frac{q(s)}{Q(s)} \tag{39}$$

$P(s),\ Q(s),\ q(s) \in \mathbb{R}[s]$ are polynomials, $a,\ b \in \mathbb{R}$ are real numbers and $r \in \mathbb{N},\ r > 1$ is an integer. A further hypothesis is that the fractions are coprime, i.e. there are no common factors between their numerators and denominators. When a combination $(s-a)Q(s)$ appears, it is understood that $Q(s)$ has no factor of $s-a$. In other words $Q(a) \neq 0$. The same considerations hold for $(s^2+as+b)Q(s)$.

To compute the full partial fraction decomposition of a given fraction, first compute the partial fraction expansion corresponding to each of the denominator roots, then sum the resulting fractions.

7.1 A simple real root

This is the simplest case, the partial fraction decomposition form for a simple real root a is

$$\frac{P(s)}{(s-a)Q(s)} = \frac{A}{s-a} + \frac{q(s)}{Q(s)}. \tag{40}$$

Multiplying both hand sides and substituting $s = a$ gives

$$A = \left.\frac{P(s)}{Q(s)}\right|_{s=a} = \frac{P(a)}{Q(a)} \tag{41}$$

7.2 A simple complex root

The partial partial fractions form for a simple complex roots is

$$\frac{P(s)}{(s^2+as+b)Q(s)} = \frac{As+B}{s^2+as+b} + \frac{q(s)}{Q(s)}. \tag{42}$$

Multiplying both sides for s^2+as+b and clearing the fractions holds

$$P(s) = (As+B)Q(s) + (s^2+as+b)q(s) \tag{43}$$

Now there are two ways to procede, the first is to substitute the two roots $z,\ \bar{z}$ of s^2+as+b in order to obtain a linear system in the unknown $A,\ B$. The second way is to observe that $s^2 = -as-b$ and to replace every occurrence of s^2 (and higher powers, if there are) with $-as-b$. In this case (43) reduces to

$$\gamma z + \delta = \alpha(A,B)z + \beta(A,B) \tag{44}$$

where $\alpha(A,B),\ \beta(A,B), \gamma,\ \delta$ are real quantities and $\alpha(A,B),\ \beta(A,B)$ depend linearly on $A,\ B$. Equating the imaginary part of the two sides gives $\gamma\mathrm{Im}(z) = \alpha(A,B)\mathrm{Im}(z)$, but the imaginary part of z is non zero, therefore $\gamma = \alpha(A,B)$. With the same argument $\delta = \beta(A,B)$. Solving that linear system permits to find $A,\ B$.

Esempio 2. *Expand*

$$\frac{s+1}{(s-1)(s^2-2s+2)} = \frac{As+B}{s^2-2s+2} + \frac{C}{s-1}. \tag{45}$$

Coefficient C is

$$C = \left.\frac{s+1}{s^2-2s+2}\right|_{s=1} = \frac{2}{1-2+2} = 2. \tag{46}$$

Now clearing the denominator of (45) leads

$$s+1 = (As+B)(s-1) + C(s^2-2s+2) = As^2 + Bs - As - B + C(s^2-2s+2). \tag{47}$$

The substitution $s^2 = 2s-2$ simplifies the expression in

$$s+1 = A(2s-2) + Bs - As - B \qquad \Rightarrow \qquad s+1 = s(2A+B-A) - 2A - B. \tag{48}$$

Equating the powers of s gives the linear system

$$A + B = 1 \qquad\qquad -2A - B = 1 \tag{49}$$

from which $A = -2$ and $B = 3$, thus the required partial fraction decomposition is

$$\boxed{\frac{s+1}{(s-1)(s^2-2s+2)} = \frac{-2s+3}{s^2-2s+2} + \frac{2}{s-1}.} \tag{50}$$

7.3 Repeated real roots

When there are repeated roots, things get more involved. The general expansion for a repeated real factor is

$$\frac{P(s)}{(s-a)^r Q(s)} = \frac{A_r}{(s-a)^r} + \frac{A_{r-1}}{(s-a)^{r-1}} + \cdots + \frac{A_1}{s-a} + \frac{q(s)}{Q(s)}. \tag{51}$$

Clearing the fractions

$$P(s) = A_r Q(s) + A_{r-1}(s-a)Q(s) + \cdots + A_1(s-a)^{r-1}Q(s) + (s-a)^r q(s) \tag{52}$$

substituting $s = a$ one has $P(a) = A_r Q(a)$. To compute A_{r-1} one differentiates (52) with respect to s, i.e

$$\begin{aligned} P'(s) &= A_r Q'(s) \\ &+ A_{r-1}\big[Q(s) + (s-a)Q'(s)\big] \\ &+ \cdots \\ &+ A_1\big[(r-1)(s-a)^{r-2}Q(s) + (s-a)^{r-1}Q'(s)\big] \\ &+ r(s-a)^{r-1}q(s) + (s-a)^r q'(s) \end{aligned} \tag{53}$$

now the substitution $s = a$ gives

$$P'(a) = A_r Q'(a) + A_{r-1}Q(a) \tag{54}$$

from which one can compute A_{r-1} because A_r is known. The coefficients $A_{r-2}, \ldots, A_1$ are computed similarly iterating the differentiation process.

Esempio 3. *Expand in partial fractions*

$$\frac{s^2+1}{(s-1)^3(s^2-2s+2)} = \frac{As+B}{s^2-2s+2} + \frac{C}{(s-1)^3} + \frac{D}{(s-1)^2} + \frac{E}{(s-1)} \tag{55}$$

C can be evaluated multiplying both sides by $(s-1)^3$ and putting $s = 1$

$$C = \left.\frac{s^2+1}{(s^2-2s+2)}\right|_{s=1} = \frac{2}{1-2+2} = 2. \tag{56}$$

Now the differentiation process begins, clearing the denominator:

$$\begin{aligned} s^2+1 &= (As+B)(s-1)^3 \\ &+ C(s^2-2s+2) \\ &+ D(s-1)(s^2-2s+2) \\ &+ E(s-1)^2(s^2-2s+2). \end{aligned} \tag{57}$$

Letting $S = 1$ remains again $C = 2$, taking the first derivative one has

$$\begin{aligned} 2s &= A(s-1)^3 + (As+B)3(s-1)^2 \\ &+ C(2s-2) \\ &+ D(s^2-2s+2) + D(s-1)(2s-2) \\ &+ 2E(s-1)(s^2-2s+2) + E(s-1)^2(2s-2). \end{aligned} \tag{58}$$

Letting $s = 1$ *remains* $2 = D$*, thus* $D = 2$*. Taking another derivative one has*

$$\begin{aligned} 2 &= 3A(s-1)^2 + (As+B)6(s-1) + 3A(s-1)^2 \\ &+ 2Cs \\ &+ D(2s-2) + D(2s-2) + 2D(s-1) \\ &+ 2E(s^2-2s+2) + 2E(s-1)(2s-2) + 2E(s-1)(2s-2) + E2(s-1)^2. \end{aligned} \tag{59}$$

Letting $s = 1$ *gives* $2 = 2C + 2E$ *thus* $E = -1$*. Now it remains the expansion of the factor of* $s^2 - 2s + 2$*, which is the case of a pair of complex conjugated roots,*

$$\begin{aligned} \frac{s^2+1}{(s-1)^3} = As+B\Big|_{s=1+\boldsymbol{i}} &\quad\Rightarrow\quad -2+\boldsymbol{i} = (1+\boldsymbol{i})A + B \\ \frac{s^2+1}{(s-1)^3} = As+B\Big|_{s=1-\boldsymbol{i}} &\quad\Rightarrow\quad -2-\boldsymbol{i} = (1-\boldsymbol{i})A + B \end{aligned} \tag{60}$$

and the solution is $A = 1$ *and* $B = -3$*. So, the desired expansion is*

$$\boxed{\frac{s^2+1}{(s-1)^3(s^2-2s+2)} = \frac{s-3}{s^2-2s+2} + \frac{2}{(s-1)^3} + \frac{2}{(s-1)^2} - \frac{1}{(s-1)}}. \tag{61}$$

7.4 Repeated complex roots

This is the most interesting and difficult case. Consider the irreducible fraction

$$\frac{P(s)}{(s^2+as+b)^r Q(s)} \tag{62}$$

with $r > 1$, where the quadratic polynomial $s^2 + as + b$ has complex roots, and where $Q(s)$ has no factor of $s^2 + as + b$. The concept of this argument is to use the auxiliary function

$$\frac{P(s)}{(s^2+as+t)Q(s)} \tag{63}$$

that admits a partial expansion of the form

$$\frac{P(s)}{(s^2+as+t)Q(s)} = \frac{A(t)s+B(t)}{s^2+as+t} + \frac{q(s,t)}{Q(s)} \tag{64}$$

where the coefficients $A(t)$ and $B(t)$ can be computed via the method described for a pair of complex roots. Then, taking the derivative with respect to t one has

$$-\frac{P(s)}{(s^2+as+t)^2Q(s)} = \frac{A'(t)s+B'(t)}{s^2+as+t} - \frac{A(t)s+B(t)}{(s^2+as+t)^2} + \frac{q_t(s,t)}{Q(s)} \qquad q_t(s,t) = \frac{\partial q}{\partial t} \tag{65}$$

Then substituting $t = b$ gives the partial expansion in the case $r = 2$. For larger r it is enough to iterate the process of differentiation. For example the case $r = 3$ yields:

$$2\frac{P(s)}{(s^2+as+t)^3Q(s)} = 2\frac{A(t)s+B(t)}{(s^2+as+t)^3} - 2\frac{A'(t)s+B'(t)}{(s^2+as+t)^2} + \frac{A''(t)s+B''(t)}{(s^2+as+t)} + \frac{q_{tt}(s,t)}{Q(s)}. \tag{66}$$

Esempio 4. *Expand in partial fractions*

$$\frac{s^2+1}{(s-1)(s^2-2s+2)^3}. \tag{67}$$

The coefficient of $s-1$ *is easily calculated*

$$\left.\frac{s^2+1}{(s^2-2s+2)^3}\right|_{s=1} = 2 \tag{68}$$

Now clearing the denominator, one has

$$s^2+1 = (A(t)s+B(t))(s-1) + q(t)(s^2-2s+t). \tag{69}$$

Simplifying terms and substituting $s^2 = 2s - t$ *gives*

$$2s - t + 1 = A(t)(2s-t) - A(t)s + B(t)s - B(t). \tag{70}$$

This gives a linear system

$$\begin{cases} 2 & = & A(t) + B(t) \\ -t+1 & = & -t\,A(t) - B(t). \end{cases} \tag{71}$$

Solving gives $A(t) = 1 - \frac{2}{t-1}$ *and* $B(t) = \frac{t+1}{t-1} = 1 + \frac{2}{t-1}$. *It is very useful to simplify the expressions for* A *and* B *in order to avoid messy derivatives. Now the derivatives of* $A(t),\ B(t)$ *are* $A'(t) = \frac{2}{(t-1)^2},\ A''(t) = -\frac{4}{(t-1)^3}$ *and* $B'(t) = -\frac{2}{(t-1)^2},\ B''(t) = \frac{4}{(t-1)^3}$. *Thus the coefficients are* $A(2) = -1,\ A'(2) = 2,\ A''(2) = -4,\ B(2) = 3,\ B'(2) = -2,\ B''(2) = 4$. *So the desired expansion is*

$$\frac{s^2+1}{(s-1)(s^2-2s+2)^3} = \frac{2}{s-1} + \frac{\frac{1}{2}2(-1s+3)}{(s^2-2s+2)^3} - \frac{\frac{1}{2}2(2s-2)}{(s^2-2s+2)^2} + \frac{\frac{1}{2}(-4s+4)}{(s^2-2s+2)} \tag{72}$$

and simplifying

$$\boxed{\frac{s^2+1}{(s-1)(s^2-2s+2)^3} = \frac{2}{s-1} - \frac{s-3}{(s^2-2s+2)^3} - \frac{2s-2}{(s^2-2s+2)^2} - \frac{2s-2}{(s^2-2s+2)}.} \tag{73}$$