Exercitation 3

Numerical Methods for Dynamical Systems and Control

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1 Exercise 1

Solve the following finite difference system of equation.

$$\begin{cases} x_{k+1} &= 1 + y_k + w_k \\ y_{k+1} &= 1 + x_k + w_k \\ w_{k+1} &= 1 + x_k + y_k \end{cases} \tag{1}$$

with $x_0 = 0,\ y_0 = 1,\ w_0 = 2.$

1.1 Solution with Z-transform

Applying the Z-transform to the system of equations one has

$$\begin{cases} zx = \dfrac{z}{z-1} + y + w \\ zy - z = \dfrac{z}{z-1} + x + w \\ zw - 2z = \dfrac{z}{z-1} + x + y. \end{cases} \tag{2}$$

To solve the linear system it is useful to write it in matrix form, putting $A = z/(z-1)$:

$$\begin{pmatrix} z & -1 & -1 & -A \\ -1 & z & -1 & -A - z \\ -1 & -1 & z & -A - 2z \end{pmatrix} \tag{3}$$

Using Gauss elimination one can reduce the matrix in echelon form. Multiplying the first row by $1/z$ and summing in the last rows,

$$\begin{pmatrix} 1 & -\frac{1}{z} & -\frac{1}{z} & -\frac{A}{z} \\ 0 & z - \frac{1}{z} & -1 - \frac{1}{z} & -A - z - \frac{A}{z} \\ 0 & -1 - \frac{1}{z} & z - \frac{1}{z} & -A - 2z - \frac{A}{z} \end{pmatrix} \tag{4}$$

Multiplying the last two rows by z and reducing,

$$\begin{pmatrix} 1 & -\frac{1}{z} & -\frac{1}{z} & -\frac{A}{z} \\ 0 & z^2-1 & -z-1 & -Az-z^2-A \\ 0 & -z-1 & z^2-1 & -Az-2z^2-A \end{pmatrix} \tag{5}$$

Multiplying the second row by $\frac{z+1}{z^2-1}$,

$$\begin{pmatrix} 1 & -\frac{1}{z} & -\frac{1}{z} & -\frac{A}{z} \\ 0 & z^2-1 & -z-1 & -A(z+1)-z^2 \\ 0 & 0 & -\frac{(z+1)^2}{z^2-1}+z^2-1 & \frac{z+1}{z^2-1}[A(z+1)-z^2]-A(z+1)-2z^2 \end{pmatrix} \tag{6}$$

Solving the last equation for w:

$$\begin{aligned} 0 &= w\left(\frac{-z-1+z^2(z-1)-z+1}{z-1}\right)-A(z+1)-2z^2-A\frac{z+1}{z-1}-\frac{z^2}{z-1} \\ &= w\left(\frac{z^3-z^2-2z}{z-1}\right)-\frac{z(z+1)}{z-1}-2z^2-\frac{z(z+1)}{(z-1)^2}-\frac{z^2}{z-1} \\ &= wz\frac{z^2-z-2}{z-1}-\frac{z(z+1)+z^2}{z-1}-2z^2-\frac{z(z+1)}{(z-1)^2} \\ &= wz\frac{(z+1)(z-2)}{z-1}-\frac{z^2+z+z^2}{z-1}-2z^2-\frac{z(z+1)}{(z-1)^2} \end{aligned} \tag{7}$$

from which

$$\begin{aligned} w\frac{z(z+1)(z-2)}{z-1} &= \frac{2z^2+z}{z-1}+2z^2-\frac{z(z+1)}{(z-1)^2} \\ &= \frac{z(2z+1)(z-1)+2z^2(z-1)^2+z(z+1)}{(z-1)^2} \\ &= \frac{2z^3-2z^2+z^2-z+2z^2(z^2-2z+1)+z^2+z}{(z-1)^2} \\ &= \frac{2z^3-z^2-z+2z^4-4z^3+2z^2+z^2+z}{(z-1)^2} \\ &= \frac{2z^4-2z^3+2z^2}{(z-1)^2} \\ &= \frac{2z^2(z^2-z+1)}{(z-1)^2} \end{aligned} \tag{8}$$

thus

$$
\begin{aligned}
w &= \frac{2z^2(z^2-z+1)}{(z-1)^2}\,\frac{z-1}{z(z+1)(z-2)} \\
&= \frac{2z^3-2z^2+2z}{z^3-2z^2-z+2} \\
&= \frac{2z(z^2-z+1)}{(z-1)(z+1)(z-2)} \\
&= \frac{\alpha z}{z-1}+\frac{\beta z}{z+1}+\frac{\gamma z}{z-2} \qquad (9)\\
&= \frac{\alpha z(z+1)(z-2)+\beta z(z-1)(z-2)+\gamma z(z^2-1)}{(z-1)(z+1)(z-2)} \\
&= \frac{\alpha(z^3-z^2-2z)+\beta(z^3-3z^2+2z)+\gamma(z^3-z)}{(z-1)(z+1)(z-2)} \\
&= \frac{z^3(\alpha+\beta+\gamma)+z^2(-\alpha-3\beta)+z(-2\alpha+2\beta-\gamma)}{(z-1)(z+1)(z-2)}
\end{aligned}
$$

Solving the associated linear system:

$$
\begin{cases} \alpha+\beta+\gamma=2 \\ -\alpha-3\beta=-2 \\ -2\alpha+2\beta-\gamma=2 \end{cases} \Rightarrow \begin{cases} \alpha=2-\beta-\gamma \\ -2+\beta+\gamma-3\beta=-2 \\ -4+2\beta+2\gamma+2\beta-\gamma=2 \end{cases} \qquad (10)
$$

$$
\begin{cases} -2\beta=-\gamma \\ 6=6\beta \end{cases} \Rightarrow \begin{cases} \alpha=-1 \\ \beta=1 \\ \gamma=2 \end{cases} \qquad (11)
$$

So, in conclusion,

$$
w=-\frac{z}{z-1}+\frac{z}{z+1}+\frac{2z}{z-2} \qquad (12)
$$

and taking the inverse Z-transform

$$
w_k=-1+(-1)^k+2\cdot 2^k \Rightarrow \boxed{w_k=2^{k+1}+(-1)^k-1.} \qquad (13)
$$

Passing to the equation in y, from (6):

$$
(z^2-1)y-(z+1)w=A(z+1)+z^2 \qquad (14)
$$

and simplifying

$$\begin{aligned}
y &= w\frac{z+1}{z^2-1} + \frac{A(z+1)}{z^2-1} + \frac{z^2}{z^2-1} \\
&= \frac{-z(z+1)(z-2) + z(z-1)(z-2) + 2z(z^2-1) + z(z+1)(z-2) + z^2(z-1)(z-2)}{(z-1)^2(z+1)(z-2)} \\
&= \frac{-z^3 + z^2 + 2z + z^3 - 3z^2 + 2z + 2z^3 - 2z + z^4 - 3z^3 + 2z^2}{(z-1)^2(z+1)(z-2)} \\
&= \frac{z^4 - z^2}{(z-1)^2(z+1)(z-2)} \\
&= \frac{z^2(z^2-1)}{(z-1)^2(z+1)(z-2)} \\
&= \frac{z^2}{(z-1)(z-2)}.
\end{aligned} \tag{15}$$

Therefore

$$y = \frac{2z}{z-2} - \frac{z}{z-1} \Rightarrow y_k = 2 \cdot 2^k - 1. \tag{16}$$

Thus the solution for y is

$$\boxed{y_k = 2^{k+1} - 1.} \tag{17}$$

The last equation to be solved is

$$\begin{aligned}
zx &= \frac{z}{z-1} + y + w \\
&= \frac{z}{z-1} + \frac{2z}{z-2} - \frac{z}{z-1} - \frac{z}{z-1} + \frac{z}{z+1} + \frac{2z}{z-2} \\
&= \frac{4z}{z-2} - \frac{z}{z-1} + \frac{z}{z+1}.
\end{aligned} \tag{18}$$

Dividing the right hand side by z

$$\begin{aligned}
x &= \frac{4}{z-2} - \frac{1}{z-1} + \frac{1}{z+1} \\
&= \frac{4(z^2-1) - (z-2)(z+1) + (z-2)(z-1)}{(z-2)(z-1)(z+1)} \\
&= \frac{4z^2 - 4 - z^2 + z + 2 + z^2 - 3z + 2}{(z-2)(z-1)(z+1)} \\
&= \frac{4z^2 - 2z}{(z-2)(z-1)(z+1)} \qquad (19) \\
&= \frac{\alpha z}{z-2} + \frac{\beta z}{z-1} + \frac{\gamma z}{z+1} \\
&= \frac{\alpha z^3 - \alpha z + \beta z^3 - \beta z^2 - 2\beta z + \gamma z^3 - 3\gamma z^2 + 2\gamma z}{(z-2)(z-1)(z+1)} \\
&= \frac{z^3(\alpha+\beta+\gamma) + z^2(-\beta-3\gamma) + z(-\alpha-2\beta+2\gamma)}{(z-2)(z-1)(z+1)}
\end{aligned}$$

Solving the associated linear system

$$\begin{cases} \alpha+\beta+\gamma = 0 \\ -\beta - 3\gamma = 4 \\ -\alpha - 2\beta + 2\gamma = -2 \end{cases} \Rightarrow \begin{cases} \alpha = -\beta-\gamma \\ -\beta = 4 + 3\gamma \\ 6\gamma = -6 \end{cases} \Rightarrow \begin{cases} \alpha = 2 \\ \beta = -1 \\ \gamma = -1 \end{cases} \qquad (20)$$

Therefore the solution for x is the inverse Z-transform of

$$x = \frac{2z}{z-2} - \frac{z}{z-1} - \frac{z}{z+1} \qquad (21)$$

which is

$$\boxed{x_k = 2^{k+1} + (-1)^{k+1} - 1.} \qquad (22)$$

□

2 Exercise 2

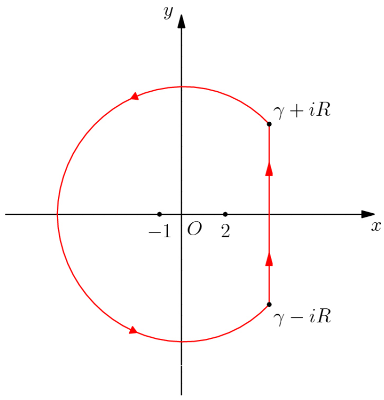
Evaluate the complex inversion integral

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+1)(s-2)^2}\right\} \tag{23}$$

by using the Bromwich's integral formula

$$f(t) = \frac{1}{2\pi \boldsymbol{i}} \int_{\gamma - \boldsymbol{i}\infty}^{\gamma + \boldsymbol{i}\infty} e^{st} F(s)\, ds. \tag{24}$$

Follow the contour of integration and the theory of residues.



2.1 Solution

The integration is to be performed along a line $s = \gamma$ in the complex plane, where γ is a real number chosen so that $s = \gamma$ lies to the right of all the singularities (in this case just poles). In practice the Bromwich's integral is evaluated by considering the contour integral

$$\frac{1}{2\pi \boldsymbol{i}} \oint_C e^{st} F(s)\, ds \tag{25}$$

where C is the path showed in the picture, and is composed of a line and an arc of a circle of radius R with center at the origin. If we represent the arc of circle by Γ, it follows that

$$f(t) = \lim_{R\to\infty} \frac{1}{2\pi \boldsymbol{i}} \int_{\gamma - \boldsymbol{i}R}^{\gamma + \boldsymbol{i}R} e^{st} F(s)\, ds = \lim_{R\to\infty} \left\{ \frac{1}{2\pi \boldsymbol{i}} \oint_C e^{st} F(s)\, ds - \frac{1}{2\pi \boldsymbol{i}} \int_\Gamma e^{st} F(s)\, ds \right\}. \tag{26}$$

If the integral around Γ approaches zero as $R \to \infty$, then by the residue theorem $f(t)$ is the sum of residues of $e^{st}F(s)$ at poles of $F(s)$.
The function of this exercise satisfies the required conditions, so

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+1)(s-2)^2}\right\} = \frac{1}{2\pi \boldsymbol{i}} \int_{\gamma - \boldsymbol{i}\infty}^{\gamma + \boldsymbol{i}\infty} \frac{e^{st}}{(s+1)(s-2)^2}\, ds = \frac{1}{2\pi \boldsymbol{i}} \oint_C \frac{e^{st}}{(s+1)(s-2)^2}\, ds \tag{27}$$

and this is equal to the sum of the residues of $\dfrac{e^{st}}{(s+1)(s-2)^2}$ at poles $s = -1$ and $s = 2$. Now, the residue at simple pole $= -1$ is

$$\lim_{s\to -1} (s+1) \frac{e^{st}}{(s+1)(s-2)^2} = \frac{1}{9} e^{-t}. \tag{28}$$

The residue at double pole $s = 2$ is

$$\begin{aligned}
\lim_{s\to 2} \frac{1}{1!}\frac{d}{ds}\left[(s-2)^2 \frac{e^{st}}{(s+1)(s-2)^2}\right] &= \lim_{s\to 2}\frac{d}{ds}\left[\frac{e^{st}}{s+1}\right] \\
&= \lim_{s\to 2}\frac{(s+1)te^{st} - e^{st}}{(s+1)^2} \qquad (29)\\
&= \frac{1}{3}te^{2t} - \frac{1}{9}e^{2t}.
\end{aligned}$$

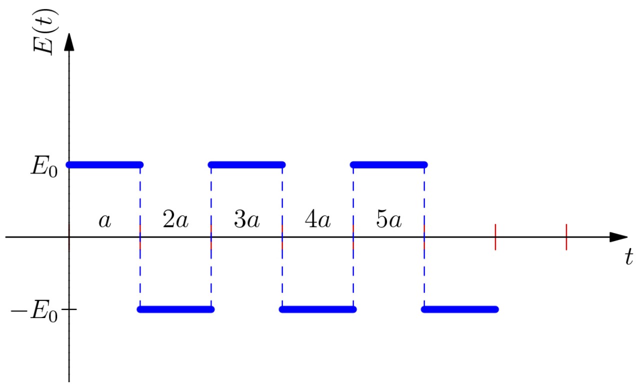
Thus

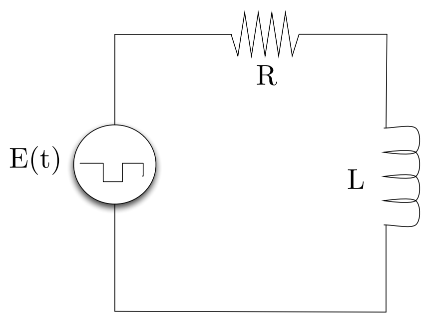
$$\mathcal{L}^{-1}\left\{\frac{1}{(s+1)(s-2)^2}\right\} = \boxed{\frac{1}{9}e^{-t} + \frac{1}{3}te^{2t} - \frac{1}{9}e^{2t}}. \qquad (30)$$

□

3 Exercise 3

A periodic voltage $E(t)$ in the form of a square wave,shown in the next figure, is applied to an electric circuit with a generator $E(t)$, a resistor R and an inductor L in series. Assuming that the current is zero at time $t = 0$, find it at any later time.





3.1 Solution

The differential equation for the current $i(t)$ in the circuit is

$$Ri(t) + L\frac{d}{dt}i(t) = E(t) \qquad (31)$$

where $i(0) = 0$. Taking the Laplace transform one has

$$RI(s) + LsI(s) = \mathcal{L}\{E(t)\} \qquad (32)$$

thus the question is to find the transform of the square wave. First there is to note that it is a periodic function of period $T = 2a$. From the property of the transform of a periodic function one has

$$F(s) = \frac{1}{1-e^{-Ts}} \int_o^T e^{-st} f(t)\, dt. \tag{33}$$

Specializing the formula for this example gives

$$\begin{aligned}
\mathcal{L}\{E(t)\} &= \frac{1}{1-e^{-2as}} \int_o^{2a} e^{-st} E(t)\, dt \\
&= \frac{1}{1-e^{-2as}} \left[\int_0^a E_0 e^{-st}\, dt - \int_a^{2a} E_0 e^{-st}\, dt \right] \\
&= \frac{E_0}{1-e^{-2as}} \left[\frac{1-e^{-as}}{s} + \frac{e^{-2as}-e^{-as}}{s} \right] \\
&= \frac{E_0}{1-e^{-2as}} \left[\frac{1-2e^{-as}+e^{-2as}}{s} \right] \\
&= \frac{E_0}{1-e^{-2as}} \frac{(1-e^{-as})^2}{s} \\
&= \frac{E_0}{s} \frac{(1-e^{-as})^2}{(1-e^{-as})(1+e^{-as})} \\
&= \frac{E_0}{s} \frac{(1-e^{-as})}{(1+e^{-as})} \\
&= \frac{E_0}{s} \tanh\left(\frac{as}{2}\right).
\end{aligned} \tag{34}$$

In fact the definition of the hyperbolic tangent is

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^x(1-e^{-2x})}{e^x(1+e^{-2x})} = \frac{(1-e^{-2x})}{(1+e^{-2x})}. \tag{35}$$

So the transformed equation becomes

$$RI(s) + LsI(s) = \frac{E_0}{s} \tanh\left(\frac{as}{2}\right) \qquad \Rightarrow \qquad I(s) = \frac{E_0}{s(Ls+R)} \tanh\left(\frac{as}{2}\right) \tag{36}$$

thus the desired solution for the current is

$$I(t) = \frac{E_0}{L} \mathcal{L}^{-1} \left\{ \frac{1}{s(s+\frac{R}{L})} \tanh\left(\frac{as}{2}\right) \right\}. \tag{37}$$

The function $F(s) = \dfrac{1}{s(s+\frac{R}{L})} \tanh\left(\dfrac{as}{2}\right)$ has a simple pole at $s = -R/L$ and simple poles at $s = s_k = \dfrac{(2k+1)\pi \boldsymbol{i}}{a}$ for $k = 0, \pm 1,\ \pm 2,\ \ldots$ where $\cosh\left(\dfrac{as}{2}\right) = 0$. The value $s = 0$ is not a pole since

$$\lim_{s\to 0}\frac{\tanh\left(\frac{as}{2}\right)}{s}=\frac{a}{2} \tag{38}$$

is finite (e.g. de l'Hopital rule), thus $s=0$ is a removable singularity. The residue of $e^{st}F(s)$ at pole $s=-R/L$ is

$$\lim_{s\to -R/L}\left(s+\frac{R}{L}\right)\frac{e^{st}}{s(s+\frac{R}{L})}\tanh\left(\frac{as}{2}\right)=\frac{L}{R}e^{-\frac{Rt}{L}}\tanh\left(\frac{aR}{2L}\right) \tag{39}$$

Residue at $s=s_k=\dfrac{(2k+1)\pi\boldsymbol{i}}{a}$ is

$$\begin{aligned}\lim_{s\to s_k}(s-s_k)\left\{\frac{e^{st}}{s(s+\frac{R}{L})}\tanh\left(\frac{as}{2}\right)\right\} &= \left\{\lim_{s\to s_k}\frac{s-s_k}{\cosh\left(\frac{as}{2}\right)}\right\}\left\{\lim_{s\to s_k}\frac{e^{st}\sinh\left(\frac{as}{2}\right)}{s(s+\frac{R}{L})}\right\} \\ &= \left\{\frac{1}{\frac{a}{2}\sinh\left(\frac{as_k}{2}\right)}\right\}\left\{\frac{e^{s_k t}\sinh\left(\frac{as_k}{2}\right)}{s_k(s_k+\frac{R}{L})}\right\} \\ &= \frac{2e^{(2k+1)\pi\boldsymbol{i}t/a}}{((2k+1)\pi\boldsymbol{i})[(2k+1)\pi\boldsymbol{i}/a+\frac{R}{L}]}\end{aligned} \tag{40}$$

The sum of the residue is

$$\frac{L}{R}e^{-\frac{Rt}{L}}\tanh\left(\frac{aR}{2L}\right)+\sum_{k=-\infty}^{\infty}\frac{2e^{(2k+1)\pi\boldsymbol{i}t/a}}{((2k+1)\pi\boldsymbol{i})[(2k+1)\pi\boldsymbol{i}/a+\frac{R}{L}]} \tag{41}$$

Now, collecting pair of term of the sum, namely $k=0$ and $k=-1$, $k=1$ and $k=-2$, and so on, using the De Moivre formula, one can rewrite the sum as

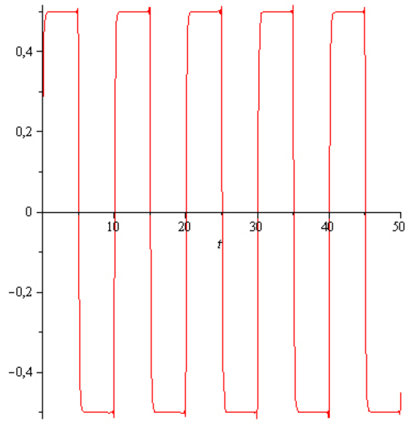
$$\frac{L}{R}e^{-\frac{Rt}{L}}\tanh\left(\frac{aR}{2L}\right)+\frac{4aL}{\pi}\sum_{n=1}^{\infty}\frac{aR\sin[(2n-1)\pi t/a]-(2n-1)\pi L\cos[(2n-1)\pi t/a]}{(2n-1)[a^2R^2+(2n-1)^2\pi^2L^2]} \tag{42}$$

Finally the required expression is

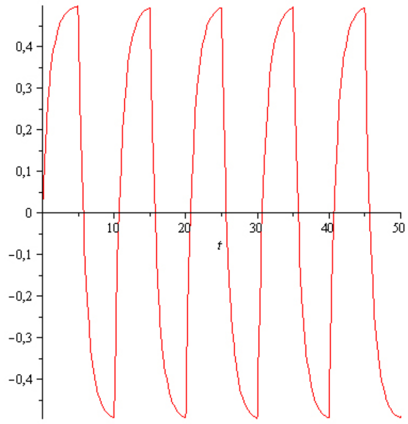
$$\boxed{i(t)=\frac{E_0}{R}e^{-\frac{Rt}{L}}\tanh\left(\frac{aR}{2L}\right)+\frac{4aE_0}{\pi}\sum_{n=1}^{\infty}\frac{aR\sin[(2n-1)\pi t/a]-(2n-1)\pi L\cos[(2n-1)\pi t/a]}{(2n-1)[a^2R^2+(2n-1)^2\pi^2L^2]}.} \tag{43}$$

In the next figures are shown the plots for $i(t)$ for three cases

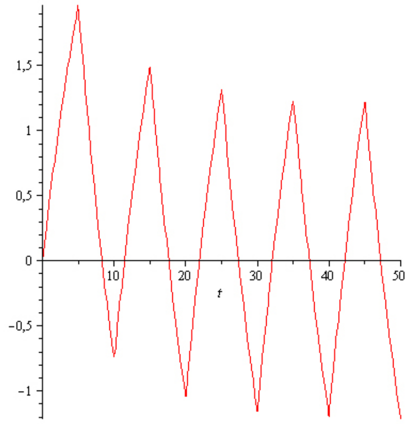
1. $i(t)$ for $E_0 = 5$, $a = 5$, $R = 10$, $L = 1$.



2. $i(t)$ for $E_0 = 5$, $a = 5$, $R = 10$, $L = 10$.



3. $i(t)$ for $E_0 = 5$, $a = 5$, $R = 1$, $L = 10$.



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