Exercitation 4

Numerical Methods for Dynamical Systems and Control

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1 Exercise 1

Solve the finite difference equation

$$y_n + y_{n-1} - 6y_{n-2} = 8u_n \tag{1}$$

with $y_{-1} = -\frac{4}{5}$ and $y_{-2} = -\frac{2}{5}$.

1.1 Solution with analytical techniques

The characteristic polynomial associated to the homogeneous equation is

$$0 = \lambda^n + \lambda^{n-1} - 6\lambda^{n-2} = \lambda^{n-2}(\lambda^2 + \lambda - 6) \Rightarrow \lambda^2 + \lambda - 6 = 0 \iff \lambda = 2, -3.$$

Both roots have multiplicity 1, so the homogeneous solution is

$$yh_n = \alpha_1 2^n + \alpha_2(-3)^n$$

for real constants $\alpha_1,\ \alpha_2$. The particular solution is $yp_n = \beta$ for a real constant β, thus substituting yp in (1) gives

$$\beta + \beta - 6\beta = 8 \Rightarrow \beta = -2.$$

The general solution will be $y_n = yh_n + yp_n = \alpha_1 2^n + \alpha_2(-3)^n + \beta$. The remaining constants can be determined from the initial conditions giving a linear system:

$$\begin{cases} \alpha_1 2^{-2} + \alpha_2(-3)^{-2} - 2 &= -\dfrac{2}{5} = y_{-2} \\ \alpha_1 2^{-1} + \alpha_2(-3)^{-1} - 2 &= -\dfrac{4}{5} = y_{-1} \end{cases}$$

$$\begin{cases} \alpha_1 \frac{1}{4} + \alpha_2 \frac{1}{9} - 2 &= -\frac{2}{5} \\ \alpha_1 \frac{1}{2} - \alpha_2 \frac{1}{3} - 2 &= -\frac{4}{5} \end{cases}$$

hence $\alpha_1 = \frac{24}{5}$ and $\alpha_2 = \frac{18}{5}$. Thus the solution of the equation is

$$\boxed{y_n = \frac{24}{5} 2^n + \frac{18}{5} (-3)^n - 2}.$$

1.2 Solution with the Z-transform

The initial conditions are not compatible with a causal function required for the Z-transform, therefore it is necessary to shift them to positive times. Another problem is that the Z-transform of y_{n-k} does not depend on the initial conditions, so the original equation should be rewritten in terms of y_{n+k}. In facts, looking at the transform of y_{n-k}:

$$\mathcal{Z}\{f_{n-k}\} = \sum_{n=0}^{\infty} f_{n-k} z^{-n}$$

now let $m = n - k$, this implies $n = m + k$, and the series becomes

$$\sum_{m=-k}^{\infty} f_m z^{-m-k} = z^{-k} \left(\sum_{m=-k}^{-1} f_m z^{-m} + \sum_{m=0}^{\infty} f_m z^{-m} \right) = z^{-k} \left(\sum_{m=-k}^{-1} f_m z^{-m} + \mathcal{Z}\{f_m\} \right).$$

Now, because the function is causal, the first sum in the parenthesis is zero, thus

$$\mathcal{Z}\{f_{n-k}\} = z^{-k} \left(\mathcal{Z}\{f_n\} .\right)$$

This is not useful in this case because of the lost of the information on the initial values. Instead the transformation

$$\mathcal{Z}\{f_{n+k}\} = z^k \left(\mathcal{Z}\{f_n\} - \sum_{m=0}^{k-1} f_m z^{-m} \right).$$

The correct way to treat this equation is to calculate $y_0,\ y_1$ starting from the initial values $y_{-2},\ y_{-1}$, and then to shift the recurrence by 2. In this case holds

$$y_n = 8u_n - y_{n-1} + 6y_{n-2}$$

thus one can obtain $y_0,\ y_1$ by recurrence:

$$y_0 = 8 + \frac{4}{5} + 6\left(-\frac{2}{5}\right) = \frac{32}{5} \qquad y_1 = 8 - \frac{32}{5} + 6\left(-\frac{4}{5}\right) = -\frac{16}{5}$$

Now one has to shift by 2 the equation forward, the substitution $m = n + 2$ is enough:

$$y_{m+2} + y_{m+1} - 6y_m = 8u_{m+2}$$

Taking the Z-transform of both sides

$$z^2 \left(Y(z) - y_0 - y_1 z^{-1}\right) + z(Y(z) - y_0) - 6Y(z) = \frac{8z}{z-1}$$

Collecting coefficients for $Y(z)$ leads to

$$Y(z)(z^2 + z - 6) = \frac{8z}{z-1} + z^2 y_0 + z y_1 + z y_0$$

this is

$$Y(z)(z-2)(z+3) = \frac{8z}{z-1} + \frac{32}{5}z^2 - \frac{16}{5}z + \frac{32}{5}z$$

and isolating $Y(z)$

$$Y(z) = \frac{8z}{(z-1)(z-2)(z+3)} + \frac{32}{5}\frac{z^2}{(z-2)(z+3)} + \frac{16}{5}\frac{z}{(z-2)(z+3)}$$

Now the necessary partial fraction decomposition is in terms of a linear combination of $\frac{z}{z \pm a}$. Usual methods used with the Laplace transform apply here with the only difference that here there is a z in the numerator. Keep in mind to simplify it when performing calculations.

$$\frac{8z}{(z-1)(z-2)(z+3)} = \frac{Az}{z-1} + \frac{Bz}{z-2} + \frac{Cz}{z+3}$$

the trick is to handle this expression as

$$\frac{8}{(z-1)(z-2)(z+3)} = \frac{A}{z-1} + \frac{B}{z-2} + \frac{C}{z+3}$$

so using standard techniques one finds out that

$$A = \frac{8}{(-1)4} = -2 \qquad B = \frac{8}{5} \qquad C = \frac{8}{(-4)(-5)} = \frac{2}{5}$$

The other two terms splits with the same technique, one is

$$\frac{32}{5}\frac{z^2}{(z-2)(z+3)} = \frac{64}{25}\frac{z}{z-2} + \frac{96}{25}\frac{z}{z+3}$$

and the last one is

$$\frac{16}{5}\frac{z}{(z-2)(z+3)} = \frac{16}{25}\frac{z}{z-2} - \frac{16}{25}\frac{z}{z+3}$$

Taking the inverse Z-transform one has the required solution:

$$y_n = -2 + \frac{8}{5}2^n + \frac{2}{5}(-3)^n + \frac{64}{25}2^n + \frac{96}{25}(-3)^n + \frac{16}{25}2^n - \frac{16}{25}(-3)^n$$

and collecting terms

$$y_n = -2 + \left(\frac{8}{5} + \frac{64}{25} + \frac{16}{25}\right)2^n + \left(\frac{2}{5} + \frac{96}{25} - \frac{16}{25}\right)(-3)^n$$

so the required solution is

$$\boxed{y_n = -2 + \frac{24}{5}2^n + \frac{18}{5}(-3)^n}.$$

□

2 Exercise 2

Write the Fourier series of the following periodic function

$$f(x) = \begin{cases} 0 & \text{if} \quad x \in \left[-\pi, -\frac{\pi}{2}\right) \\ \sin(x) & \text{if} \quad x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \\ 0 & \text{if} \quad x \in \left[\frac{\pi}{2}, \pi\right) \end{cases}$$

2.1 Solution with the classic formula

The classic formulas for the Fourier's series of a periodic function f are

$$a_n = \frac{2}{T}\int_T f(x)\cos(n\omega x)\,dx \qquad b_n = \frac{2}{T}\int_T f(x)\sin(n\omega x)\,dx \qquad \omega = \frac{2\pi}{T}$$

where T is the period of the function and ω the frequency of oscillation. The function f can be written as the trigonometric series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(n\omega x) + b_n\sin(n\omega x).$$

In this exercise $T = 2\pi$ so $\omega = 1$. The integral for a_n is

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin(x)\cos(nx)\,dx = 0$$

for all $n \geq 0$ because the integrand is an odd function on a symmetric interval. For b_n one has

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin(x)\sin(nx)\,dx.$$

To solve this integral use the Euler's formula

$$\cos(x) = \frac{e^{\boldsymbol{i}x} + e^{-\boldsymbol{i}x}}{2} \qquad \sin(x) = \frac{e^{\boldsymbol{i}x} - e^{-\boldsymbol{i}x}}{2\boldsymbol{i}}$$

thus the integral for b_n becomes

$$\begin{aligned}
b_n &= \frac{1}{\pi}\frac{1}{(2\boldsymbol{i})^2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(e^{\boldsymbol{i}x} - e^{-\boldsymbol{i}x}\right)\left(e^{\boldsymbol{i}nx} - e^{-\boldsymbol{i}nx}\right)\, dx \\
&= -\frac{1}{4\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{\boldsymbol{i}x(n+1)} - e^{-\boldsymbol{i}x(n-1)} - e^{\boldsymbol{i}x(n-1)} + e^{-\boldsymbol{i}x(n+1)}\, dx \\
&= -\frac{1}{4\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\left(\cos((n+1)x) - \cos((n-1)x)\right)\, dx \\
&= -\frac{1}{\pi}\int_{0}^{\frac{\pi}{2}} \left(\cos((n+1)x) - \cos((n-1)x)\right)\, dx \\
&= -\frac{1}{\pi}\int_{0}^{\frac{\pi}{2}} \frac{1}{n+1}\cos(y)\, dy + \frac{1}{\pi}\int_{0}^{\frac{\pi}{2}} \frac{1}{n-1}\cos(y)\, dy \\
&= -\frac{1}{\pi}\frac{1}{n+1}\left[\sin((n+1)x)\right]_0^{\frac{\pi}{2}} + \frac{1}{\pi}\frac{1}{n-1}\left[\sin((n-1)x)\right]_0^{\frac{\pi}{2}} \\
&= -\frac{1}{\pi(n+1)}\sin((n+1)\frac{\pi}{2}) + \frac{1}{\pi(n-1)}\sin((n-1)\frac{\pi}{2})
\end{aligned}$$

Now observing that $\sin((n-1)\frac{\pi}{2}) = -\sin((n+1)\frac{\pi}{2}) = \cos(n\frac{\pi}{2})$ the integral reduces to

$$\begin{aligned}
b_n &= -\frac{1}{\pi(n+1)}\sin((n+1)\frac{\pi}{2}) - \frac{1}{\pi(n-1)}\sin((n+1)\frac{\pi}{2}) \\
&= \frac{-(n-1)\sin((n+1)\frac{\pi}{2}) - (n+1)\sin((n+1)\frac{\pi}{2})}{\pi(n^2-1)} \\
&= -\frac{2n\sin((n+1)\frac{\pi}{2})}{\pi(n^2-1)}
\end{aligned}$$

and collecting terms, at the end

$$a_n = 0 \qquad b_n = -\frac{2n\cos(n\frac{\pi}{2})}{\pi(n^2-1)}.$$

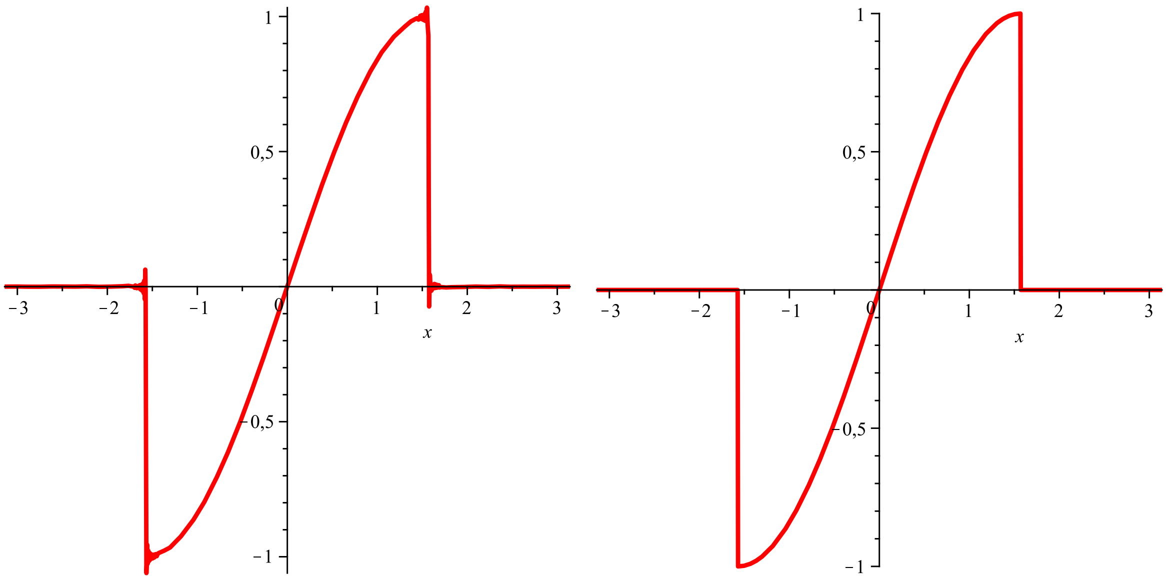

Figure 1: On the left $f(x)$ truncated series with 500 terms, on the right $f(x)$ with the piece-wise definition

Therefore f can be written as

$$\boxed{f(x) = -\sum_{n=1}^{\infty} \frac{2n\cos(n\frac{\pi}{2})}{\pi(n^2-1)}}\,.$$

Notice that the summand is not singular for $n = 1$, because $b_1 = \frac{1}{2}$.

2.2 Solution with the complex formula

The expansion in Fourier series in with complex exponentials is

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i\omega n x} \qquad c_n = \frac{1}{T}\int_T f(x)e^{-in\omega x}\,dx$$

So the computation of c_n is

$$
\begin{aligned}
c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-\boldsymbol{i}n\omega x}\,dx \\
&= \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin(x)e^{-\boldsymbol{i}n\omega x}\,dx \\
&= \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{e^{\boldsymbol{i}x}-e^{-\boldsymbol{i}x}}{2\boldsymbol{i}}\right) e^{-\boldsymbol{i}n\omega x}\,dx \\
&= \frac{1}{2\pi}\,\frac{1}{2\boldsymbol{i}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{\boldsymbol{i}x(n-1)}-e^{-\boldsymbol{i}x(n+1)}\,dx \\
&= \frac{1}{2\pi}\,\frac{1}{2\boldsymbol{i}}\left(\frac{e^{-\boldsymbol{i}\frac{\pi}{2}(n-1)}-e^{\boldsymbol{i}\frac{\pi}{2}(n-1)}}{-\boldsymbol{i}(n-1)}-\frac{e^{-\boldsymbol{i}\frac{\pi}{2}(n+1)}-e^{\boldsymbol{i}\frac{\pi}{2}(n+1)}}{-\boldsymbol{i}(n+1)}\right) \\
&= \frac{1}{2\pi}\,\frac{1}{2\boldsymbol{i}}\left(\frac{-2\boldsymbol{i}\sin((n-1)\frac{\pi}{2})}{-\boldsymbol{i}(n-1)}-\frac{-2\boldsymbol{i}\sin((n+1)\frac{\pi}{2})}{-\boldsymbol{i}(n+1)}\right) \\
&= \frac{1}{2\pi}\,\frac{1}{2\boldsymbol{i}}\left(\frac{2\boldsymbol{i}\sin((n+1)\frac{\pi}{2})}{-\boldsymbol{i}(n-1)}+\frac{2\boldsymbol{i}\sin((n+1)\frac{\pi}{2})}{-\boldsymbol{i}(n+1)}\right) \\
&= \frac{1}{2\pi}\,\frac{1}{2\boldsymbol{i}}\left(\frac{2\boldsymbol{i}(n+1)\sin((n+1)\frac{\pi}{2})+2\boldsymbol{i}(n-1)\sin((n+1)\frac{\pi}{2})}{-\boldsymbol{i}(n^2-1)}\right) \\
&= \frac{\boldsymbol{i}}{2\pi}\frac{2n\sin((n+1)\frac{\pi}{2})}{(n^2-1)}.
\end{aligned}
$$

The real trigonometric form can be retrieved observing that $c_0 = 0$ and collecting together the terms for n and $-n$.

2.3 Solution in series of cosines

The fourier series can be written as an expansion of only cosines (sines). The transformation from the classical formula to this (devoted to signal processing) is the following.

$$
f(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos\left(\frac{2\pi n x}{T} - \phi_n\right)
$$

where $A_0 = \frac{a_0}{2}$ and for $n > 0$ $A_n = \frac{1}{2}\sqrt{a_n^2 + b_n^2}$, $\phi_n = \arctan\frac{b_n}{a_n}$.

□

3 Exercise 3

Calculate the solution of this system of recurrences.

$$\begin{aligned} x_{k+1} &= x_k + k \\ y_{k+1} &= y_k + w_k \\ w_{k+1} &= x_k + y_k + w_k \end{aligned}$$

with initial conditions $x_0 = 0,\ y_0 = 0,\ w_0 = 1$.

3.1 Solution with Z-transform

The transformed system of equation becomes

$$\begin{aligned} zX - zx_0 &= X + \frac{z}{(z-1)^2} \\ zY - zy_0 &= Y + W \\ zW - zw_0 &= X + Y + W. \end{aligned}$$

Substituting the ICS and collecting term the system reduces to

$$\begin{aligned} X &= \frac{z}{(z-1)^3} \\ W &= Y(z-1) \\ X + z &= Y[(z-1)^2 - 1]. \end{aligned}$$

Now focusing only on the first equation, the solution for x_k can be found on the tables of Z-transform, in fact

$$\mathcal{Z}\left\{a^k \binom{k}{j}\right\} = \frac{a^n z}{(z-a)^{j+1}} \Rightarrow \mathcal{Z}^{-1}\left\{\frac{z}{(z-1)^3}\right\} = \binom{k}{2} = \frac{1}{2}k(k-1)$$

thus

$$\boxed{x_k = \frac{1}{2}k(k-1)}\,.$$

The third equation gives the relation for y_k, i.e.

$$X + z = Y[(z-1)^2 - 1] \qquad \Rightarrow \qquad Y = \frac{z}{z(z-1)^3(z-2)} + \frac{z}{z(z-2)}$$

simplifying the previous expression leads to

$$Y = \frac{z + z(z-1)^3}{z(z-1)^3(z-2)} = \frac{z(z^2 - 3z + 3)}{(z-1)^3(z-2)}.$$

To expand this expression for Y in partial fractions, it is useful to consider the same equation but in the form Y/z, therefore

$$\frac{Y}{z} = \frac{(z^2-3z+3)}{(z-1)^3(z-2)} = \frac{A}{z-2} + \frac{B}{z-1} + \frac{C}{(z-1)^2} + \frac{D}{(z-1)^3} \tag{2}$$

The coefficients A and D are easy to establish via direct substitution, $A = \frac{4-6+3}{1} = 1$ and $D = \frac{1-3+3}{-1} = -1$. The best way to obtain B, C is to multiply (2) by z and then let $z \to \infty$.

$$\lim_{z\to\infty} \frac{Yz}{z} = \lim_{z\to\infty} \frac{z(z^2-3z+3)}{(z-1)^3(z-2)} = 1 + B = 0$$

hence $B = -1$. Now the substitution $z = 0$ in (2) leads to

$$\frac{3}{(-2)(-1)} = -\frac{1}{2} - (-1) + C + 1 \Rightarrow C = 0.$$

In conclusion, the partial fraction decomposition of (2) is

$$Y = \frac{z}{z-2} - \frac{z}{z-1} - \frac{z}{(z-1)^3}$$

thus the inverse Z-transform is

$$\boxed{y_k = 2^k - 1 - \frac{1}{2}k(k-1)}\,.$$

It remains the equation for W, from the second equation of the initial system one has

$$W = Y(z-1) = \frac{z(z^2-3z+3)}{(z-1)^2(z-2)}.$$

Performing partial fraction decomposition

$$\frac{W}{z} = \frac{(z^2-3z+3)}{(z-1)^2(z-2)} = \frac{A}{z-2} + \frac{B}{z-1} + \frac{C}{(z-1)^2}$$

and as before $A = 1$, $C = -1$, multiplying by$z$ and letting $z \to \infty$

$$\lim_{z\to\infty} \frac{Wz}{z} = 1 + B = 1 \qquad \Rightarrow B = 0.$$

Then the solution of the recurrence in w is the inverse Z-transform of

$$W = \frac{z}{z-2} - \frac{z}{(z-1)^2}$$

i.e.

$$\boxed{w_k = 2^k - k}\,.$$

□