Exercitation 5

Numerical Methods for Dynamical Systems and Control

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1 Exercise 1

Solve this system of differential equations.

$$y''(t) + y'(t) = e^t$$
$$x''(t) - y'(t) = t$$

with initial conditions $x(0) = 1$, $x'(0) = 0$, $y(0) = -1$ and $y'(0) = 1$.

1.1 Solution with ODE techniques

Consider the first equation which is independent of $x(t)$. The solution of a second order linear differential equation with constant coefficient is the sum of the homogeneous equation $y_h(t)$ plus the particular solution $y_p(t)$. The characteristic polynomial associated to the homogeneous equation is $\lambda^2 + \lambda = 0$, thus its solutions are $\lambda = 0, -1$. Hence the homogeneous equation has the form

$$y_h(t) = c_1 + c_2 e^{-t}$$

for real constants $c_1,\ c_2$. The particular solution has the form $y_p(t) = \alpha e^t$ for a real constant α. The general solution in therefore

$$y(t) = c_1 + c_2 e^{-t} + \alpha e^t.$$

Calculating $y''(t) + y'(t) = e^t$ and equating the coefficients, gives $\alpha = \frac{1}{2}$. The substitution of the initial conditions, leads to the linear system for the coefficients $c_1,\ c_2$. They are $c_1 = -1$ and $c_2 = -\frac{1}{2}$. So the solution of the first differential equation is

$$\boxed{y(t) = -1 + \frac{1}{2}(e^t - e^{-t})}\,.$$

The second equation of the system involves $x''(t)$, using the knowledge of $y(t)$ one can substitute in the equation the derivative of $y(t)$ and integrate twice to obtain $x(t)$.

$$x''(t) - y'(t) - t = x''(t) - \frac{1}{2}e^{t} - \frac{1}{2}e^{-t} - t$$

Isolating $x''(t)$ one has

$$x''(t) = t + \frac{1}{2}e^{t} + \frac{1}{2}e^{-t} \quad \Rightarrow \quad x'(t) = \int_0^t z + \frac{1}{2}e^{z} + \frac{1}{2}e^{-z}\, dz + c$$

solving the integral, leads

$$x'(t) = \frac{t^2}{2} + \frac{1}{2}e^{t} - \frac{1}{2}e^{-t} + c \Rightarrow x'(0) = c = 0$$

thus $c = 0$ and performing another step of integration

$$x(t) = \int_0^t \frac{x^2}{2} + \frac{1}{2}e^{x} - \frac{1}{2}e^{-x}\, dx + c = \frac{t^3}{6} + \frac{1}{2}e^{t} + \frac{1}{2}e^{-t} + c.$$

Imposing the initial conditions one finds $c = 0$, thus the solution is

$$\boxed{x(t) = \frac{t^3}{6} + \frac{1}{2}e^{t} + \frac{1}{2}e^{-t}}\,.$$

1.2 Solution with Laplace transform

The transformed system is

$$\begin{aligned} s^2Y - sy(0) - y'(0) + sY - y(0) &= \frac{1}{s-1} \\ s^2X - sx(0) - x'(0) - sY + y(0) &= \frac{1}{s^2} \end{aligned}$$

substituting the ICS and collecting terms it becomes

$$\begin{aligned} s^2Y - s + sY \quad &= \frac{1}{s-1} \\ s^2X - s - sY - 1 &= \frac{1}{s^2} \end{aligned}$$

From the first equation, one can solve for Y

$$Y(s^2 + s) = \frac{1}{s-1} - s \Rightarrow Y = \frac{1}{s(s+1)(s-1)} - \frac{s}{s(s+1)}$$

the reduction to partial fractions is easily done via substitution because there are distinct poles with no multiplicity.

$$Y = -\frac{1}{s} + \frac{\frac{1}{2}}{s+1} + \frac{\frac{1}{2}}{s-1} - \frac{1}{s+1} = -\frac{1}{s} - \frac{\frac{1}{2}}{s+1} + \frac{\frac{1}{2}}{s-1}$$

and the antitransform is

$$\boxed{y(t) = -1 + \frac{1}{2}(e^t - e^{-t})}\,.$$

Now using the knowledge of the transform of Y from the second equation of the system one gets

$$s^2 X = \frac{1}{s^2} + s + 1 + s\left(-\frac{1}{s} - \frac{\frac{1}{2}}{s+1} + \frac{\frac{1}{2}}{s-1}\right)$$

from which

$$X = \frac{1}{s^4} + \frac{1}{s} - \frac{\frac{1}{2}}{s(s+1)} + \frac{\frac{1}{2}}{s(s-1)}$$

and taking the least common multiple[1]

$$\begin{aligned} X &= \frac{s^2 - 1 + s^3(s^2-1) - \frac{1}{2}s^3(s-1) + \frac{1}{2}s^3(s+1)}{s^4(s+1)(s-1)} \\ &= \frac{s^2 - 1 + s^5 - s^3 - \frac{1}{2}s^4 + \frac{1}{2}s^3 + \frac{1}{2}s^4 + \frac{1}{2}s^3}{s^4(s+1)(s-1)} \\ &= \frac{s^5 + s^2 - 1}{s^4(s+1)(s-1)} \\ &= \frac{A}{s^4} + \frac{B}{s^3} + \frac{C}{s^2} + \frac{D}{s} + \frac{E}{s+1} + \frac{F}{s-1}. \end{aligned}$$

Coefficients A, E, F are easy to determine just by substitution:

$$A = \frac{-1}{-1} = 1 \qquad E = \frac{-1+1-1}{1(-2)} = \frac{1}{2} \qquad F = \frac{1}{2}.$$

Multiplying both sides by s and letting $s \to \infty$ gives $1 = D + E + F \Rightarrow 1 = D + 1$ therefore $D = 0$. Now the shortest way of get B and C is to impose the passage for two arbitrary points, e.g. $s = \pm 2$ and solve the associated linear system.

$$\lim_{s\to 2} X = \frac{32+4-1}{48} = \frac{1}{16} + \frac{B}{8} + \frac{C}{4} + \frac{\frac{1}{2}}{3} + \frac{1}{2} \Rightarrow \frac{35}{48} = \frac{3 + 6B + 12C + 8 + 24}{48}$$

thus there is the first relation $B = -2C$. The second limit yields

$$\lim_{s\to -2} X = \frac{-32+4-1}{48} = \frac{1}{16} - \frac{B}{8} + \frac{C}{4} + \frac{\frac{1}{2}}{-1} + \frac{\frac{1}{2}}{3} \Rightarrow \frac{-29}{48} = \frac{8 + 24C - 24 - 8}{48}$$

[1] If one looks carefully at the Laplace transform tables, finds directly the antitransform of $\frac{\alpha}{s(s+\alpha)}$ which is $1 - e^{-\alpha t}$. Performing the lcm is interesting because of the pole of fourth order in 0 in the partial fraction decomposition.

thus $C = B = 0$. The required expansion in partial fraction is

$$X = \frac{1}{s^4} + \frac{\frac{1}{2}}{s+1} + \frac{\frac{1}{2}}{s-1}$$

and the inverse Laplace transform is

$$\boxed{x(t) = \frac{t^3}{6} + \frac{1}{2}e^t + \frac{1}{2}e^{-t}}.$$

□

2 Exercise 2

Solve Fibonacci's recurrence

$$f_{n+2} = f_{n+1} + f_n$$

with initial conditions $f_0 = f_1 = 1$.

2.1 Solution with Z-transform

Applying the Z-transform to both sides of the equation one has

$$z^2F - f_0z^2 - f_1z = zF - f_0z + F$$

collecting terms

$$F = \frac{f_0z^2 + (f_1 - f_0)z}{z^2 - z - 1}$$

and substituting the initial values

$$F = \frac{z^2}{z^2 - z - 1}. \tag{1}$$

The roots of $z^2 - z - 1$ are $z_{1,2} = \dfrac{1 \pm \sqrt{5}}{2}$, thus the expansion is

$$\frac{F}{z} = \frac{z}{z^2 - z - 1} = \frac{A}{z - z_1} + \frac{B}{z - z_2}.$$

By the method of substitution one has

$$A = \frac{\frac{1+\sqrt{5}}{2}}{\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}} = \frac{\frac{1+\sqrt{5}}{2}}{\sqrt{5}} = \frac{\sqrt{5}+5}{10}.$$

The coefficient B is equal to the conjugate[2] of A, that is $B = \frac{\sqrt{5}+5}{10}$. Hence the Fibonacci's recurrence is

$$\begin{aligned} f_n &= Az_1^n + Bz_2^n \\ &= \frac{5+\sqrt{5}}{10}\left(\frac{1+\sqrt{5}}{2}\right)^n + \frac{5-\sqrt{5}}{10}\left(\frac{1-\sqrt{5}}{2}\right)^n \\ &= \boxed{\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}} \end{aligned}$$

the first elements are the famous 1, 1, 2, 3, 5, 8, 13, 21 . . . .

2.2 Solution with complex analysis

Equation (1) can be inverted to time domain via the complex integral inversion formula for the Z-transform.

$$f_n = \oint_C F(z)z^{n-1}\,dz = \oint_C \frac{z^2 \cdot z^{n-1}}{z^2 - z - 1}\,dz = \mathrm{Res}(z_1) + \mathrm{Res}(z_2)$$

The residues are respectively

$$\begin{aligned} \mathrm{Res}(z_1) &= \lim_{z\to z_1}(z-z_1)\frac{z^2 \cdot z^{n-1}}{z^2-z-1} = \lim_{z\to z_1}\frac{z^{n+1}}{z-z_2} = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} \\ \mathrm{Res}(z_2) &= \lim_{z\to z_2}(z-z_2)\frac{z^2 \cdot z^{n-1}}{z^2-z-1} = \lim_{z\to z_2}\frac{z^{n+1}}{z-z_1} = -\frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1} \end{aligned}$$

□

3 Exercise 3

Solve the following system of recurrences.

$$\begin{aligned} x_{k+2} &= k + y_k \\ y_{k+1} &= 2y_k \end{aligned}$$

with $x_0 = 0,\ x_1 = 1,\ y_0 = 1.$

[2]conjugate over $\mathbb{Q}(\sqrt{5})$ which is the quadratic estension of $\mathbb{Q}$

3.1 Solution with Z-transform

The transformed system is

$$
\begin{aligned}
z^2X - z &= \frac{z}{(z-1)^2} + Y \\
zY - z &= 2Y
\end{aligned}
$$

thus from the second equation one has $Y = \frac{z}{z-2}$ and the reverse transform yields $\boxed{y_k = 2^k}$. The substitution of the second equation in the first gives

$$
z^2X - z = \frac{z}{(z-1)^2} + \frac{z}{z-2}
$$

and collecting terms

$$
X = \frac{1}{z} + \frac{z}{z(z-1)^2} + \frac{z}{z(z-2)}. \tag{2}
$$

If one takes the least common multiple has

$$
X = \frac{(z-1)^2(z-2) + z - 2 + (z-1)^2}{z(z-1)^2(z-2)} = \frac{z^3 - 3z^2 + 4z - 3}{z(z-1)^2(z-2)}. \tag{3}
$$

The reduction in partial fractions is

$$
X = \frac{z^3 - 3z^2 + 4z - 3}{z(z-1)^2(z-2)} = \frac{Az}{z} + \frac{Ez}{z^2} + \frac{Bz}{z-1} + \frac{Cz}{(z-1)^2} + \frac{Dz}{z-2}.
$$

With the usual trick one considers the expression X/z,

$$
\frac{X}{z} = \frac{z^3 - 3z^2 + 4z - 3}{z^2(z-1)^2(z-2)} = \frac{A}{z} + \frac{E}{z^2} + \frac{B}{z-1} + \frac{C}{(z-1)^2} + \frac{D}{z-2}.
$$

The coefficients E, C, D can be calculated via substitution:

$$
E = \frac{-3}{-2} = \frac{3}{2} \qquad C = \frac{1-3+4-3}{1(-1)} = 1 \qquad D = \frac{8-12+8-3}{4} = \frac{1}{4}\,,
$$

the remaining two can be calculated using the limit $z \to \infty$ and a direct substitution of an arbitrary point (e.g. $z = -1$).

$$
\lim_{z\to\infty} \frac{Xz}{z} = \lim_{z\to\infty} \frac{z^3 - 3z^2 + 4z - 3}{z(z-1)^2(z-2)} = A + B + D = 0,
$$

thus there is a first relation $A + B = -\frac{1}{4}$. The limit for $z \to -1$ gives the second one:

$$
\lim_{z\to-1} \frac{X}{z} = \lim_{z\to-1} \frac{z^3 - 3z^2 + 4z - 3}{z^2(z-1)^2(z-2)} = -A - \frac{B}{2} + \frac{C}{4} - \frac{D}{3} + E = \frac{-1-3-4-3}{4(-3)}
$$

that is, using the first relation ($A = -B - \frac{1}{4}$),

$$\frac{11}{12} = B + \frac{1}{4} - \frac{B}{2} + \frac{1}{4} - \frac{1}{12} + \frac{3}{2} \qquad \Rightarrow \qquad B = -2,\ A = \frac{7}{4}$$

In conclusion, the partial fraction decomposition of (3) is

$$X = \frac{7}{4} + \frac{3}{2z} + \frac{-2z}{z-1} + \frac{z}{(z-1)^2} + \frac{z}{4(z-2)}.$$

Applying the inverse Z-transform one obtains the required solution

$$\boxed{x_k = \frac{7}{4}\delta_0 + \frac{3}{2}\delta_1 - 2 + k + \frac{2^k}{4}}.$$

3.2 Solution with complex analysis

Suppose to restart from equation (3) and use the theory of residues, let call

$$F(z) = \frac{z^3 - 3z^2 + 4z - 3}{z(z-1)^2(z-2)} \quad \Rightarrow \quad f_k = \oint_C F(z) z^{k-1}\, dz = \sum \text{Res}$$

and the residues are respectively in 0, 1, 2:

$$\begin{aligned} \text{Res}(F,0) &= \lim_{z\to 0} zF(z)z^{k-1} = 0 \\ \text{Res}(F,2) &= \lim_{z\to 2} (z-2)F(z)z^{k-1} = \frac{2^k}{4}. \end{aligned}$$

In $z = 1$ there is a double pole,

$$\text{Res}(F,1) = \frac{1}{1!} \lim_{z\to 2} \frac{d}{dz}\left((z-1)^2 F(z) z^{k-1}\right)$$

this residue gives

$$\begin{aligned} \text{Res}(F,1) &= \lim_{z\to 1} \frac{[(k+2)z^{k+1} - 3(k+1)z^k + 4kz^{k-1} - 3(k-1)z^{k-2}](z^2-2z)}{(z^2-2z)^2} \\ &\lim_{z\to 1} \frac{-[z^{k+2} - 3z^{k+1} + 4z^k - 3z^{k-1}](2z-2)}{(z^2-2z)^2} \\ &= -\frac{[(k+2) - 3(k+1) + 4k - 3(k-1)] - 0}{(-1)^2} \\ &= k - 2. \end{aligned}$$

So, for $k \geq 2$

$$x_k = \frac{2^k}{4} + k - 2.$$

In order to avoid some more calculation, one could also start from (2) to decompose in partial fraction.

□

4 Exercise 4

Solve the following system of recurrences.

$$\begin{aligned} x_{k+1} + 2y_{k+1} &= x_k \\ x_{k+1} - 2y_{k+1} &= y_k \end{aligned}$$

with $x_0 = y_0 = 1$.

4.1 Solution with Z-transform

The transformed system becomes

$$\begin{aligned} zX - z + 2zY - 2z = X \\ zX - z - 2zY + 2z = Y \end{aligned}$$

Now the addition of the two equations, and the subtraction of the second from the first give

$$\begin{aligned} 2zX - 2z &= X + Y & \qquad X &= \frac{2z}{2z-1} + \frac{Y}{2z-1} \\ && \Rightarrow && \\ 4zY - 4z &= X - Y & \qquad Y(4z+1) &= 4z + \frac{2z}{2z-1} + \frac{Y}{2z-1} \end{aligned}$$

Solving the second one for Y yields

$$Y\left(4z + 1 - \frac{1}{2z-1}\right) = \frac{4z(2z-1) + 2z}{2z-1} \Rightarrow Y\left(\frac{8z^2 - 4z + 2z - 1 - 1}{2z-1}\right) = \frac{8z^2 - 4z + 2z}{2z-1},$$

thus

$$Y = \frac{8z^2 - 2z}{8z^2 - 2z - 2} = \frac{z\left(z - \frac{1}{4}\right)}{z^2 - \frac{1}{4}z - \frac{1}{4}} = \frac{Az}{z - z_1} + \frac{Bz}{z - z_2}$$

where z_1, z_2 are the roots of $z^2 - \frac{1}{4}z - \frac{1}{4} = 0$, i.e.

$$z_{1,2} = \frac{\frac{1}{4} \pm \sqrt{\frac{1}{16} + 1}}{2} = \frac{\frac{1}{4} \pm \frac{\sqrt{17}}{4}}{2} = \frac{1}{8} \pm \frac{\sqrt{17}}{8}.$$

Coefficients A, B can be calculated by substitution, giving

$$A = \frac{\frac{1}{8} + \frac{\sqrt{17}}{8} - \frac{1}{4}}{\frac{1}{8} + \frac{\sqrt{17}}{8} - \left(\frac{1}{8} - \frac{\sqrt{17}}{8}\right)} = \left(\frac{\sqrt{17}}{8} - \frac{1}{8}\right)\left(\frac{8}{2\sqrt{17}}\right) = \frac{1}{2} - \frac{1}{2\sqrt{17}}\frac{\sqrt{17}}{\sqrt{17}} = \frac{1}{2} - \frac{\sqrt{17}}{34},$$

hence $B = \frac{1}{2} + \frac{\sqrt{17}}{34}$. Therefore the solution is

$$\boxed{y_k = Az_1^k + Bz_2^k = \frac{1}{2}\left(z_1^k + z_2^k\right) - \frac{\sqrt{17}}{34}\left(z_1^k - z_2^k\right)}. \tag{4}$$

The same calculation done with the trick of multiplying by z and letting $z \to \infty$ joined with the substitution of $z = 0$ in the expression of Y/z gives

$$\lim_{z\to\infty} \frac{z\left(z - \frac{1}{4}\right)}{z^2 - \frac{1}{4}z - \frac{1}{4}} = \frac{Az}{z - z_1} + \frac{Bz}{z - z_2} \Rightarrow \qquad 1 = A + B$$
$$\lim_{z\to 0} \frac{\left(z - \frac{1}{4}\right)}{z^2 - \frac{1}{4}z - \frac{1}{4}} = \frac{A}{z - z_1} + \frac{B}{z - z_2} \Rightarrow 1 = \frac{A}{-z_1} + \frac{B}{-z_2}.$$

Solving the system gives the same solution but this way seems more involved than the direct substitution. The last method is with the complex residues,

$$F(z) = \frac{z\left(z - \frac{1}{4}\right)}{z^2 - \frac{1}{4}z - \frac{1}{4}} \quad \Rightarrow \quad f_k = \oint_C F(z)z^{k-1}\, dz = \sum \text{Res},$$

$$\text{Res}(F, z_1) = \lim_{z\to z_1}(z - z_1)F(z)z^{k-1} = \frac{z_1^k(z_1 - \frac{1}{4})}{z_1 - z_2},$$
$$\text{Res}(F, z_2) = \lim_{z\to z_2}(z - z_2)F(z)z^{k-1} = \frac{z_2^k(z_2 - \frac{1}{4})}{z_2 - z_1}.$$

This method gives directly the reverse Z-transform and does not involve calculation until the last passage, so

$$y_k = \frac{z_1^k(z_1 - \frac{1}{4})}{z_1 - z_2} + \frac{z_2^k(z_2 - \frac{1}{4})}{z_2 - z_1},$$

doing the simplification gives the (4). Passing to the equation for X, one has

$$\begin{aligned} X &= \frac{z}{z - \frac{1}{2}} + \frac{Az}{2(z - z_1)(z - \frac{1}{2})} + \frac{Bz}{2(z - z_2)(z - \frac{1}{2})} \\ &= \frac{z}{z - \frac{1}{2}} + \frac{A}{2}\frac{1}{z_1 - \frac{1}{2}}\frac{z}{z - z_1} + \frac{A}{2}\frac{1}{\frac{1}{2} - z_1}\frac{z}{z - \frac{1}{2}} + \frac{B}{2}\frac{1}{z_2 - \frac{1}{2}}\frac{z}{z - z_2} + \frac{B}{2}\frac{1}{\frac{1}{2} - z_2}\frac{z}{z - \frac{1}{2}} \end{aligned}$$

Here it is enough to compute one coefficient, the others are easily related. So the first coefficient is

$$
\begin{aligned}
\frac{A}{2}\frac{1}{z_1-\frac{1}{2}} &= \left(\frac{1}{4}-\frac{\sqrt{17}}{2\cdot 34}\right)\left(\frac{1}{\frac{1}{8}+\frac{\sqrt{17}}{8}-\frac{1}{2}}\right) \\
&= \left(\frac{1}{4}-\frac{\sqrt{17}}{2\cdot 34}\right)\left(\frac{1}{-\frac{3}{8}+\frac{\sqrt{17}}{8}}\right) \\
&= \left(\frac{1}{4}-\frac{\sqrt{17}}{2\cdot 34}\right)\frac{8}{-3+\sqrt{17}} = \frac{1}{17}\frac{34-2\sqrt{17}}{-3+\sqrt{17}} \\
&= \frac{2}{17}\frac{17-\sqrt{17}}{-3+\sqrt{17}}\frac{-3-\sqrt{17}}{-3-\sqrt{17}} \\
&= \frac{2}{17}\frac{-51+3\sqrt{17}-17\sqrt{17}+17}{9-17} = \frac{2}{17}\frac{-34-14\sqrt{17}}{-8} \\
&= \frac{1}{2}+\frac{7}{34}\sqrt{17}.
\end{aligned}
$$

Therefore the remaining coefficient are the conjugate in $\mathbb{Q}(\sqrt{17})$ and/or with opposite sign:

$$
\begin{aligned}
\frac{B}{2}\frac{1}{z_2-\frac{1}{2}} &= \frac{1}{2}-\frac{7}{34}\sqrt{17} \\
\frac{A}{2}\frac{1}{\frac{1}{2}-z_1} &= -\frac{1}{2}-\frac{7}{34}\sqrt{17} \\
\frac{B}{2}\frac{1}{\frac{1}{2}-z_2} &= -\frac{1}{2}+\frac{7}{34}\sqrt{17}
\end{aligned}
$$

Performing the inverse Z-transform one has

$$
\begin{aligned}
x_k = {}& \left(\frac{1}{2}\right)^k + \left(\frac{1}{2}+\frac{7}{34}\sqrt{17}\right)z_1^k + \left(-\frac{1}{2}-\frac{7}{34}\sqrt{17}\right)\left(\frac{1}{2}\right)^k \\
&+ \left(\frac{1}{2}-\frac{7}{34}\sqrt{17}\right)z_2^k + \left(-\frac{1}{2}+\frac{7}{34}\sqrt{17}\right)\left(\frac{1}{2}\right)^k.
\end{aligned}
$$

If one collects terms obtains

$$
\boxed{x_k = \left(\frac{1}{2}+\frac{7}{34}\sqrt{17}\right)z_1^k + \left(\frac{1}{2}-\frac{7}{34}\sqrt{17}\right)z_2^k}.
$$

It is easy to check that $x_0 = 1$ as required.

□