Exercitation 6

Numerical Methods for Dynamical Systems and Control

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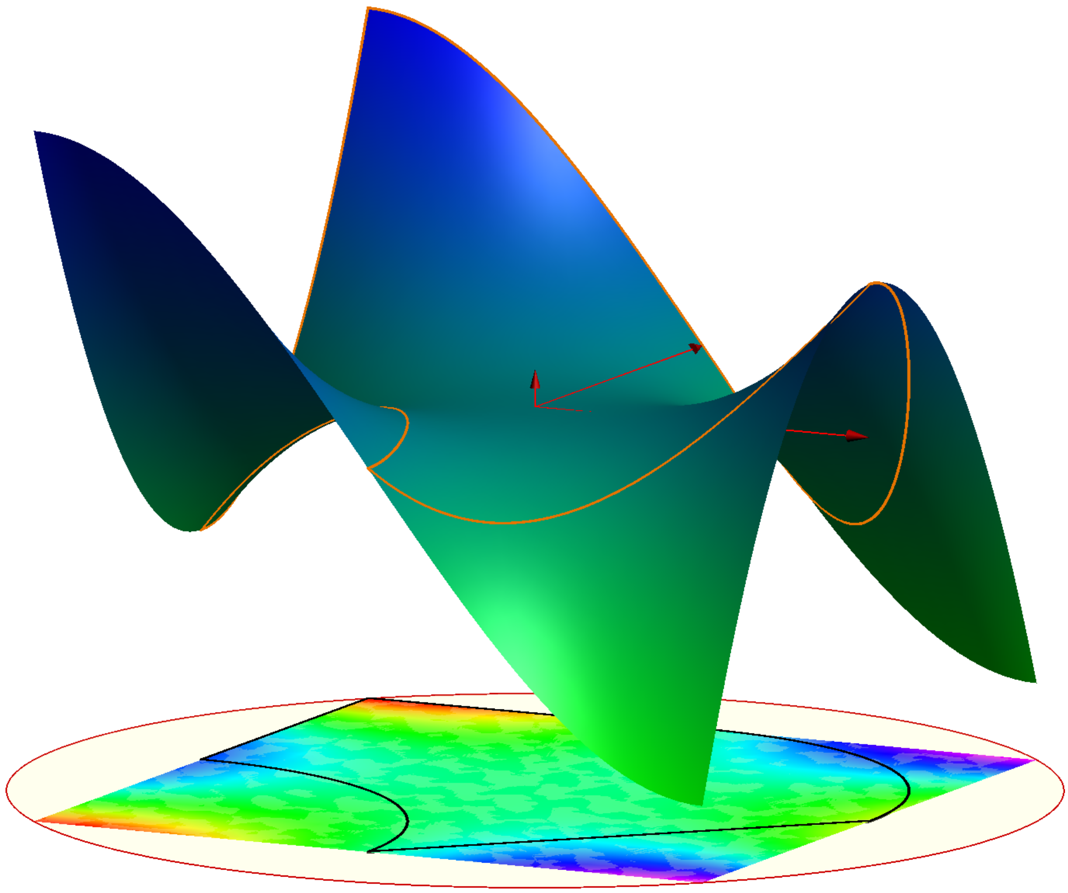
November 16, 2011

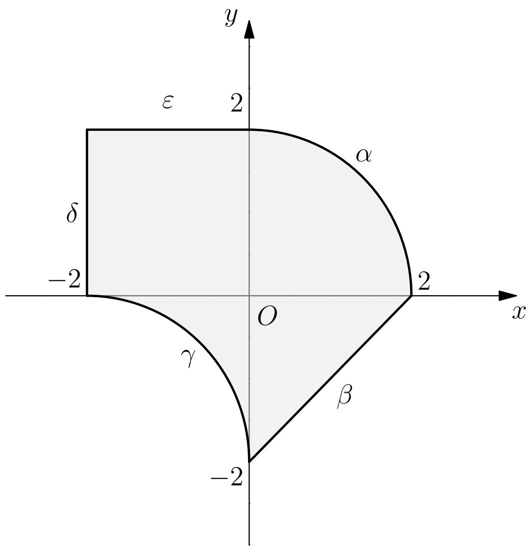
1 Exercise 1

Find maxima and minima of the function $f : \mathbb{R}^2 \to \mathbb{R}$, *the monkey saddle*, restricted to the domain Ω (in black in the next picture, in orange on the surface below).

$$f(x, y) = x^3 - 3xy^2$$



Description of Ω. The border of the domain is made up of straight lines and circle arcs, it is a subset of the square $[-2,2] \times [-2,2]$.



1.1 Solution with calculus

The first thing to do is to find stationary points solving the Jacobian equal to zero.

$$\nabla f = \begin{pmatrix} 3x^2 - 3y^2 \\ -6y \end{pmatrix}^T = 0 \iff x = y = 0.$$

The nature of point $O = (0,0)$ can be discovered by the study of the Hessian matrix,

$$\nabla^2 f(0,0) = \begin{pmatrix} 6x & -6y \\ 0 & -6 \end{pmatrix}\Bigg|_{x=y=0} = \begin{pmatrix} 0 & 0 \\ 0 & -6 \end{pmatrix}.$$

It turns out that $\nabla^2 f$ is negative semidefinite in O, so this test is inconclusive. However, because the search of maxima and minima is performed on a compact set and f is continuous, it is enough to compare the value of f in O with the maxima and minima of f on the border of the domain. One has that $f(O) = 0$.
The parametrization of path α is $y_\alpha(x) = +\sqrt{4 - x^2}$ for $x \in [0,2]$, so on the surface it becomes

$$f(x, y_\alpha(x)) = 4x^3 - 12x.$$

At the extrema holds $f(A_1) = f(0,2) = 0$ and $f(A_3) = f(2,0) = 8$, on the arc the stationary points are obtained when the gradient is zero, i.e.

$$\frac{\partial}{\partial x} f(x, y_\alpha(x)) = \frac{\partial}{\partial x}(4x^3 - 12x) = 12x^2 - 12 = 0 \iff x = \pm 1.$$

The solution $x = -1$ is not in the domain, so it has to be discarded. For $x = 1$ the stationary point is $A_2 = (1, \sqrt{3})$, the value of f there is $f(1, \sqrt{3}) = -8$.
The parametrization of path β is $y_\beta(x) = x - 2$ for $x \in [0,2]$, so on the surface it becomes

$$f(x, y_\beta(x)) = -2x^3 + 12x^2 - 12x.$$

At the extrema holds $f(B_1) = f(0,-2) = 0$ and $f(B_2) = f(2,0) = 8$, on the segment the stationary points are obtained when the gradient is zero, i.e.

$$\frac{\partial}{\partial x} f(x, y_\beta(x)) = \frac{\partial}{\partial x}(-2x^3 + 12x^2 - 12x) = -6x^2 + 24x - 12 = 0 \iff x = 2 \pm \sqrt{2}.$$

The solution $x = 2 + \sqrt{2}$ is not in the domain, so it has to be discarded. For $x = 2 - \sqrt{2}$ the stationary point is $B_3 = (2 - \sqrt{2}, -\sqrt{2})$, the value of f there is

$$f(B_3) = f(2 - \sqrt{2}, -\sqrt{2}) = (2 - \sqrt{2})^3 - 12 + 6\sqrt{2} = 8(1 - \sqrt{2}) \approx -3.313.$$

The parametrization of path γ is $y_\gamma(x) = \sqrt{4 - (x+2)^2} - 2$ for $x \in [-2, 0]$, so on the surface it becomes

$$f(x, y_\gamma(x)) = 4x^3 + 12x^2 - 12x + 12x\sqrt{4 - (x+2)^2}.$$

At the extrema holds $f(C_1) = f(0,-2) = 0$ and $f(C_3) = f(-2,0) = -8$, on the segment the stationary points are obtained when the gradient is zero, i.e.

$$\begin{aligned} \frac{\partial}{\partial x} f(x, y_\gamma(x)) &= \frac{\partial}{\partial x}(4x^3 + 12x^2 - 12x + 12x\sqrt{4 - (x+2)^2}) \\ &= \frac{12[(x^2 + 2x - 1)\sqrt{4 - (x+2)^2} - 2x^2 - 6x]}{\sqrt{4 - (x+2)^2}}. \end{aligned}$$

The elimination of the radical in the numerator permits to solve $\dfrac{\partial}{\partial x} f(x, y_\gamma(x)) = 0$. This is not an easy task. Consider

$$\frac{\partial}{\partial x} f(x, y_\gamma(x)) = 0 \iff x^6 + 8x^5 + 22x^4 + 28x^3 + 21x^2 + 4x = 0$$

The analysis of this equation is not trivial. One solution is $x = 0$ which gives the point C_1 treated before as an extrem, so the division of the sextic by x yields

$$\frac{\partial}{\partial x} f(x, y_\gamma(x)) = 0 \iff x^5 + 8x^4 + 22x^3 + 28x^2 + 21x + 4 = 0.$$

The study of this quintic polynomial is not the point of the exercise and requires some advanced results of algebra, it is enough to say that it has only one real root in the interval $[-2, 0]$. This root can be easily found after factoring the quintic in

$$x^5 + 8x^4 + 22x^3 + 28x^2 + 21x + 4 = (x^2 + 4x + 1)(x^3 + 4x^2 + 5x + 4).$$

The required solution is one of the roots of the quadratic factor $x^2 + 4x + 1$, namely $x = -2 + \sqrt{3} \approx -0.27$. Hence the stationary point inside γ is $C_2 = (-2 + \sqrt{3}, -1)$, the function evaluated there gives $f(C_2) = -20 + 12\sqrt{3} \approx 0.785$.
The parametrization of path δ is $x_\delta(y) = -2$ for $y \in [0, 2]$, so on the surface it becomes

$$f(x_\delta(y), y) = (-2)^3 + 6y^2 = 6y^2 - 8.$$

At the extrema holds $f(D_1) = f(-2, 0) = -8$ and $f(D_2) = f(-2, 2) = 16$, on the segment the stationary points are obtained when the gradient is zero, i.e.

$$\frac{\partial}{\partial y} f(x(y), y) = \frac{\partial}{\partial y}(6y^2 - 8) = 12y = 0 \iff y = 0.$$

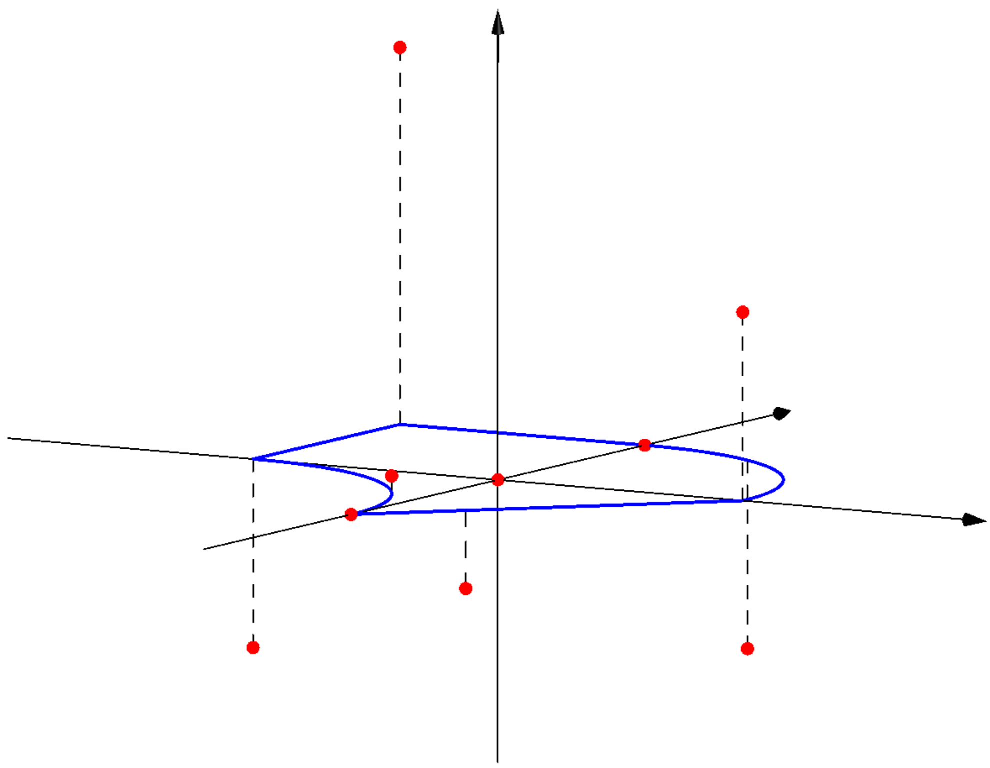
The solution $y = 0$ gives the point (already considered) $D_1 = (-2, 0) = C_3$.
The parametrization of path ε is $y_\varepsilon(x) = 2$ for $x \in [-2, 0]$, so on the surface it becomes

$$f(x, y_\varepsilon(x)) = x^3 - 12x.$$

At the extrema holds $f(E_1) = f(D_2) = f(-2, 2) = 16$ and $f(E_2) = f(A_1) = f(0, 2) = 0$, on the segment the stationary points are obtained when the gradient is zero, i.e.

$$\frac{\partial}{\partial x} f(x, y(x)) = \frac{\partial}{\partial x}(x^3 - 12x) = 3x^2 - 12 = 0 \iff x = \pm 2.$$

The solution $x = 2$ is outside ε so there is just the case $x = -2$ which is again an extremum and coincides with E_1.
To get the maximum and minimum of f in the domain Ω it is enough to compare the values of f over the critical points discovered so far. Having a look to the picture is straightforward to say that f reaches its maximum 16 in $D_2 = E_1 = (-2, 2)$, and has minimum -8 in $C_3 = D_1 = (-2, 0)$ and in $A_2 = (1, \sqrt{3})$.



2 Exercise 2

Solve the following constrained optimization.

$$f(x, y, z) = x^2 + y^2 + z^2$$

with constraints

$$h_1(x, y, z) = xy + yz + xz - 1 = 0 \qquad h_2(x, y, z) = x - y = 0.$$

2.1 Solution with Lagrange multipliers

First compute the Lagrangian,

$$\begin{aligned} \mathcal{L}(x, y, z, \lambda, \mu) &= f(x, y, z) - \lambda h_1(x, y, z) - \mu h_1(x, y, z) \\ &= x^2 + y^2 + z^2 - \lambda(xy + yz + xz - 1) - \mu(x - y) \end{aligned}$$

Then there is to solve the gradient of $\mathcal{L}$ equal to zero, i.e.

$$\nabla\mathcal{L}(x, y, z, \lambda, \mu) = \begin{pmatrix} \frac{\partial \mathcal{L}}{\partial x} \\ \frac{\partial \mathcal{L}}{\partial y} \\ \frac{\partial \mathcal{L}}{\partial z} \\ \frac{\partial \mathcal{L}}{\partial \lambda} \\ \frac{\partial \mathcal{L}}{\partial \mu} \end{pmatrix}^T = \begin{pmatrix} 2x - \lambda(y + z) - \mu \\ 2y - \lambda(x + z) + \mu \\ 2z - \lambda(y + x) \\ xy + yz + xz - 1 \\ x - y \end{pmatrix}^T = 0$$

This is a non linear system, from the last equation one has $x = y$, that should be substituted in the other equations. Equation $\frac{\partial \mathcal{L}}{\partial \lambda}$ yields

$$\frac{\partial \mathcal{L}}{\partial \lambda} = x^2 + 2xz - 1 = 0 \implies z = \frac{1 - x^2}{2x} \quad x \neq 0.$$

Putting this expression for z in $\frac{\partial \mathcal{L}}{\partial z}$ returns

$$\frac{\partial \mathcal{L}}{\partial z} = 2\frac{1 - x^2}{2x} - \lambda 2x = 0 \implies \lambda = \frac{1 - x^2}{2x^2}.$$

The knowledge of $x = y$, $z = \frac{1-x^2}{2x}$ and $\lambda = \frac{1-x^2}{2x^2}$ in the first and second equation of the gradient of $\mathcal{L}$ gives

$$\frac{\partial \mathcal{L}}{\partial x} = 2x - \frac{(1-x^2)}{2x^2}\left(x + \frac{1-x^2}{2x}\right) - \mu = 0$$
$$\frac{\partial \mathcal{L}}{\partial y} = 2x - \frac{(1-x^2)}{2x^2}\left(x + \frac{1-x^2}{2x}\right) + \mu = 0.$$

Eliminating μ for example summing the two equation gives

$$\frac{\partial \mathcal{L}}{\partial x} + \frac{\partial \mathcal{L}}{\partial y} = 4x - \frac{(1-x^2)}{x^2}\left(x + \frac{1-x^2}{2x}\right) = 0$$

Therefore simplifying the expression

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial x} + \frac{\partial \mathcal{L}}{\partial y} &= 4x - \frac{(1-x^2}{x^2}\left(\frac{x^2+1}{2x}\right) \\ &= 4x + \frac{x}{2} - \frac{1}{2x^3} \\ &= \frac{8x^4-1}{2x^3} \\ &= \frac{9x^4-1}{2x^3} \\ &= \frac{(3x^2-1)(3x^2+1)}{2x^3}\end{aligned}$$

The four roots of this equation are $\pm\frac{\sqrt{3}}{3}$ and $\pm\boldsymbol{i}\frac{\sqrt{3}}{3}$. The complex solution have to be discarded. By backward substitution one can obtain the solution of the nonlinear system, in particular

$$\frac{\partial \mathcal{L}}{\partial x} = 0 \implies \mu = 2x - \frac{(1-x^2)}{2x^2}\left(x + \frac{1-x^2}{2x}\right) = 2x - \frac{1-x^4}{4x^3}\bigg|_{x^2=1/3} = 0.$$

The same occurs with the equation $\dfrac{\partial \mathcal{L}}{\partial y} = 0$, so in both cases $\mu = 0$. Now from $\lambda = \frac{1-x^2}{2x^2}$,

$$\lambda = \frac{1-x^2}{2x^2}\bigg|_{x^2=1/3} = \frac{1-1/3}{2/3} = 1.$$

Finally, from $x = y$ and $z = \frac{1-x^2}{2x}$,

$$y = x\Big|_{x^2=1/3} = \pm\frac{\sqrt{3}}{3} \qquad z = \pm\frac{\sqrt{3}}{3}.$$

In conclusion the two stationary points are

$$P_1 = \left\{ x = \frac{\sqrt{3}}{3}, y = \frac{\sqrt{3}}{3}, z = \frac{\sqrt{3}}{3}, \lambda = 1, \mu = 0 \right\}$$

$$P_2 = \left\{ x = -\frac{\sqrt{3}}{3}, y = -\frac{\sqrt{3}}{3}, z = -\frac{\sqrt{3}}{3}, \lambda = 1, \mu = 0 \right\}.$$

Now discuss the nature of those points, one needs the Hessian matrix for f.

$$\nabla^2 f = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

No further analysis is required, both points are minima.
One can check that $\nabla^2 \mathcal{L}$ projected in the kernel of the gradient of the constraints is positive defined.
A way to identify the nature of the stationary points is to check the definition of $\nabla^2 \mathcal{L}$ in the kernel of the gradient of the constraints. So first compute the gradient and the Hessian of the constraints,

$$\nabla \boldsymbol{H} = \begin{pmatrix} \nabla h_1 \\ \nabla h_2 \end{pmatrix} = \begin{pmatrix} y+z & x+z & x+y \\ 1 & -1 & 0 \end{pmatrix}$$

$$\nabla^2 h_1 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \qquad \nabla^2 h_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The gradient evaluated in P_1 and P_2 gives respectively

$$\nabla \boldsymbol{H}(P_1) = \begin{pmatrix} \frac{2\sqrt{3}}{3} & \frac{2\sqrt{3}}{3} & \frac{2\sqrt{3}}{3} \\ 1 & -1 & 0 \end{pmatrix} \qquad \nabla \boldsymbol{H}(P_2) = \begin{pmatrix} -\frac{2\sqrt{3}}{3} & -\frac{2\sqrt{3}}{3} & -\frac{2\sqrt{3}}{3} \\ 1 & -1 & 0 \end{pmatrix}.$$

These are two pairs of linear independent vectors, thus the two kernels will have dimension 1. Hence the two bases of the kernels are $w_1 = (1, 1, -2)$ for P_1 and $w_2 = (1, 1, -2) = w_1$ for P_2.
In order to analyse the definition of the Hessian of the Lagrangian, first compute $\nabla^2 \mathcal{L}$,

$$\nabla^2 \mathcal{L} = \nabla^2 f - \lambda \nabla^2 h_1 - \mu \nabla^2 h_2.$$

Evaluating it in the two points gives

$$\nabla^2 \mathcal{L}(P_1) = \nabla^2 \mathcal{L}(P_2) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} - \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} - 0 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

Using Sylvester's theorem on $\nabla^2 \mathcal{L}$ one can not conclude anything on the nature of the points, in facts the three minors (m_1, m_2, m_3) of $\nabla^2 \mathcal{L}$ give respectively

$$m_1 = 2 \qquad m_2 = 3 \qquad m_3 = 0.$$

But to find the nature of the stationary points it is enough now to perform

$$w_1^T \nabla^2 \mathcal{L}(P_1) w_1 = (1,1,-2) \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = 18 > 0$$

hence P_1 and P_2 are minima.

3 Exercise 3

Solve the following constrained optimization.

$$f(x,y,z) = xyz$$

with constraints

$$h_1(x,y,z) = xy + yz + xz - 1 = 0 \qquad h_2(x,y,z) = x - y = 0.$$

3.1 Solution with Lagrange multipliers

First compute the Lagrangian,

$$\begin{aligned} \mathcal{L}(x,y,z,\lambda,\mu) &= f(x,y,z) - \lambda h_1(x,y,z) - \mu h_1(x,y,z) \\ &= xyz - \lambda(xy + yz + xz - 1) - \mu(x - y) \end{aligned}$$

Then there is to solve the gradient of $\mathcal{L}$ equal to zero, i.e.

$$\nabla \mathcal{L}(x,y,z,\lambda,\mu) = \begin{pmatrix} \dfrac{\partial \mathcal{L}}{\partial x} \\ \dfrac{\partial \mathcal{L}}{\partial y} \\ \dfrac{\partial \mathcal{L}}{\partial z} \\ \dfrac{\partial \mathcal{L}}{\partial \lambda} \\ \dfrac{\partial \mathcal{L}}{\partial \mu} \end{pmatrix}^T = \begin{pmatrix} yz - \lambda(y+z) - \mu \\ xz - \lambda(x+z) + \mu \\ xy - \lambda(y+x) \\ xy + yz + xz - 1 \\ x - y \end{pmatrix}^T = 0$$

This is a non linear system, from the last equation one has $x = y$, that should be substituted in the other equations. Equation $\dfrac{\partial \mathcal{L}}{\partial \lambda}$ yields

$$\frac{\partial \mathcal{L}}{\partial \lambda} = x^2 + 2xz - 1 = 0 \implies z = \frac{1 - x^2}{2x} \quad x \neq 0.$$

Putting this expression for z in $\dfrac{\partial \mathcal{L}}{\partial z}$ returns

$$\frac{\partial \mathcal{L}}{\partial z} = x^2 - \lambda 2x = 0 \implies \lambda = \frac{x^2}{2x} \implies \lambda = \frac{x}{2}.$$

The knowledge of $x = y$, $z = \frac{1-x^2}{2x}$ and $\lambda = \frac{x}{2}$ in the first and second equation of the gradient of $\mathcal{L}$ gives

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial x} &= x\frac{(1-x^2)}{2x} - \frac{x}{2}\left(x + \frac{1-x^2}{2x}\right) - \mu = 0\\ \frac{\partial \mathcal{L}}{\partial y} &= x\frac{(1-x^2)}{2x} - \frac{x}{2}\left(x + \frac{1-x^2}{2x}\right) + \mu = 0.\end{aligned}$$

Eliminating μ for example summing the two equation gives

$$\frac{\partial \mathcal{L}}{\partial x} + \frac{\partial \mathcal{L}}{\partial y} = 2x\frac{(1-x^2}{2x} - x\left(x + \frac{1-x^2}{2x}\right) = 0$$

Therefore simplifying the expression

$$1 - x^2 - x^2 - \frac{1}{2} + \frac{x^2}{2} = 0 \implies 3x^2 = 1 \implies x = \pm\sqrt{1/3} = \pm\frac{\sqrt{3}}{3}.$$

By backward substitution one can obtain the solution of the nonlinear system, in particular

$$\frac{\partial \mathcal{L}}{\partial x} = 0 \implies \mu = x\frac{(1-x^2)}{2x} - \frac{x}{2}\left(x + \frac{1-x^2}{2x}\right) = \frac{x^2}{2} - \frac{1-x^2}{4} = \frac{3x^2-1}{4}\Big|_{x^2=1/3} = 0.$$

The same occurs with the equation $\dfrac{\partial \mathcal{L}}{\partial y} = 0$, so in both cases $\mu = 0$. Now from $\lambda = \dfrac{x}{2}$,

$$\lambda = \frac{x}{2}\Big|_{x^2=1/3} = \pm\frac{\sqrt{3}}{6}.$$

Finally, from $x = y$ and $z = \frac{1-x^2}{2x}$,

$$y = x\Big|_{x^2=1/3} = \pm\frac{\sqrt{3}}{3} \qquad z = \pm\frac{\sqrt{3}}{3}.$$

In conclusion the two stationary points are

$$\begin{aligned}P_1 &= \left\{x = \frac{\sqrt{3}}{3}, y = \frac{\sqrt{3}}{3}, z = \frac{\sqrt{3}}{3}, \lambda = \frac{\sqrt{3}}{6}, \mu = 0\right\}\\ P_2 &= \left\{x = -\frac{\sqrt{3}}{3}, y = -\frac{\sqrt{3}}{3}, z = -\frac{\sqrt{3}}{3}, \lambda = -\frac{\sqrt{3}}{6}, \mu = 0\right\}.\end{aligned}$$

Now discuss the nature of those points, one needs the Hessian matrix for f and h_1, h_2.

$$\nabla^2 f = \begin{pmatrix} 0 & z & y \\ z & 0 & x \\ y & x & 0 \end{pmatrix}\Bigg|_{P_1,P_2} = \pm\frac{\sqrt{3}}{2}\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

$$\nabla^2 h_1 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \qquad \nabla^2 h_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Using Sylvester's theorem on $\nabla^2 f$ one can not conclude anything on the nature of the points, in facts the three minors (m_1, m_2, m_3) of $\nabla^2 f$ give respectively

$$m_1 = 0 \qquad m_2 = \pm\frac{\sqrt{3}}{2}(-1) = \mp\frac{\sqrt{3}}{2} \qquad m_3 = \pm\frac{\sqrt{3}}{2}\det\nabla^2 f = \pm\frac{\sqrt{3}}{2}2$$

so the Hessian is not defined. So the correct way to identify the nature of the stationary points is to check the definition of $\nabla^2\mathcal{L}$ in the kernel of the gradient of the constraints. So first compute the gradient of the constraints,

$$\nabla\boldsymbol{H} = \begin{pmatrix} \nabla h_1 \\ \nabla h_2 \end{pmatrix} = \begin{pmatrix} y+z & x+z & x+y \\ 1 & -1 & 0 \end{pmatrix}$$

The gradient evaluated in P_1 and P_2 gives respectively

$$\nabla\boldsymbol{H}(P_1) = \begin{pmatrix} \frac{2\sqrt{3}}{3} & \frac{2\sqrt{3}}{3} & \frac{2\sqrt{3}}{3} \\ 1 & -1 & 0 \end{pmatrix} \qquad \nabla\boldsymbol{H}(P_2) = \begin{pmatrix} -\frac{2\sqrt{3}}{3} & -\frac{2\sqrt{3}}{3} & -\frac{2\sqrt{3}}{3} \\ 1 & -1 & 0 \end{pmatrix}.$$

These are two pairs of linear independent vectors, thus the two kernels will have dimension 1. Hence the two bases of the kernels are $w_1 = (1, 1, -2)$ for P_1 and $w_2 = (1, 1, -2) = w_1$ for P_2.
In order to analyse the definition of the Hessian of the Lagrangian, first compute $\nabla^2\mathcal{L}$,

$$\nabla^2\mathcal{L} = \nabla^2 f - \lambda\nabla^2 h_1 - \mu\nabla^2 h_2.$$

Evaluating it in the two points gives

$$\nabla^2\mathcal{L}(P_1) = \frac{\sqrt{3}}{3}\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} - \frac{\sqrt{3}}{6}\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} - 0\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \frac{\sqrt{3}}{6}\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

$$\nabla^2\mathcal{L}(P_2) = -\frac{\sqrt{3}}{3}\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} + \frac{\sqrt{3}}{6}\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} - 0\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = -\frac{\sqrt{3}}{6}\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

To find the nature of the stationary points it is enough now to perform

$$w_1^T\nabla^2\mathcal{L}(P_1)w_1 = (1, 1, -2)\frac{\sqrt{3}}{6}\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = -\sqrt{3} < 0$$

hence P_1 is a maximum, and

$$w_2^T \nabla^2 \mathcal{L}(P_2) w_2 = (1,1,-2)\frac{-\sqrt{3}}{6}\begin{pmatrix}0 & 1 & 1\\ 1 & 0 & 1\\ 1 & 1 & 0\end{pmatrix}\begin{pmatrix}1\\ 1\\ -2\end{pmatrix} = \sqrt{3} > 0$$

hence P_2 is a minimum.