Exercitation 7

Numerical Methods for Dynamical Systems and Control

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1 Exercise 1

Solve the following constrained minimization

$$f(x,y) = x(y-1)$$

subject to

$$x^2 + y^2 \le 1 \qquad x + \frac{1}{2} \le y.$$

1.1 Solution with KKT

First put the constraints in the form of the theorem, i.e. $g_i(x,y) \ge 0$, this gives

$$g_1(x,y) = 1 - x^2 - y^2 \ge 0 \qquad g_2(x,y) = y - x - \frac{1}{2} \ge 0.$$

The next step is to build the Lagrangian,

$$\begin{aligned}\mathcal{L}(x,y,\mu_1,\mu_2) &= f(x,y) - \mu_1 g_1(x,y) - \mu_2 g_2(x,y) \\ &= x(y-1) - \mu_1(1 - x^2 - y^2) - \mu_2\left(y - x - \frac{1}{2}\right).\end{aligned}$$

So using the (first order) KKT conditions, the associated non linear system is

$$\begin{aligned}&\nabla_{(x,y)}\mathcal{L}(x,y,\mu_1,\mu_2)^T = \mathbf{0} \\ &\mu_1 g_1(x,y) = 0 \\ &\mu_2 g_2(x,y) = 0\end{aligned}$$

with the conditions

$$\mu_1 \ge 0 \qquad \mu_2 \ge 0 \qquad g_1 \ge 0 \qquad g_2 \ge 0.$$

The gradient of the Lagrangian is

$$\nabla_{(x,y)}\mathcal{L}(x,y,\mu_1,\mu_2) = \begin{pmatrix} \dfrac{\partial \mathcal{L}}{\partial x} \\ \\ \dfrac{\partial \mathcal{L}}{\partial y} \end{pmatrix}^T = \begin{pmatrix} y - 1 + 2\mu_1 x + \mu_2 \\ x + 2\mu_1 y - \mu_2 \end{pmatrix}^T .$$

In other words one has to solve the following non linear system

$$\begin{aligned} y - 1 + 2\mu_1 x + \mu_2 &= 0 \\ x + 2\mu_1 y - \mu_2 &= 0 \\ \mu_1(1 - x^2 - y^2) &= 0 \\ \mu_2\left(y - x - \frac{1}{2}\right) &= 0 \end{aligned}$$

This system is quite complex, so it is better to split it and solve it in several steps. First put $\mu_1 = 0$ and solve the simplified system, then put $\mu_2 = 0$ and solve, and so on.

- $\mu_1 = 0$. Setting $\mu_1 = 0$ yields to the simpler system

$$\begin{aligned} y - 1 + \mu_2 &= 0 \\ x - \mu_2 &= 0 \\ \mu_2\left(y - x - \frac{1}{2}\right) &= 0. \end{aligned}$$

  From the second equation one has $x = \mu_2$, thus it remains a system of two equations in two unknown.

$$\begin{cases} y + x - 1 & = \ 0 \\ x\left(y - x - \dfrac{1}{2}\right) & = \ 0 \end{cases} \implies \begin{cases} y = 1 - x \\ x\left(\dfrac{1}{2} - 2x\right) = 0 \end{cases}$$

  hence $x = 0$ or $x = \frac{1}{4}$. From x one can obtain $y = 1$ or $y = \frac{3}{4}$ and $\mu_2 = 0$ or $\mu_2 = \frac{1}{4}$. In conclusion there are two solutions in this case, namely

$$\begin{aligned} P_1 &= \{x = 0, y = 1, \mu_1 = 0, \mu_2 = 0\} \\ P_2 &= \left\{x = \frac{1}{4}, y = \frac{3}{4}, \mu_1 = 0, \mu_2 = \frac{1}{4}\right\} \end{aligned}$$

- $\mu_2 = 0$. Setting $\mu_2 = 0$ leads to the simpler system

$$\begin{aligned} y - 1 + 2\mu_1 x &= 0 \\ x + 2\mu_1 y &= 0 \\ \mu_1\left(1 - x^2 - y^2\right) &= 0. \end{aligned}$$

From the second equation one has $x = -2\mu_1 y$, thus it remains a system of two equations in two unknown.

$$\begin{cases} y - 1 - 4\mu_1^2 y & = 0 \\ \mu_1 \left(1 - 4\mu_1^2 y^2 - y^2\right) & = 0 \end{cases} \implies y = \frac{1}{1 - 4\mu_1^2}.$$

It remains a single expression for μ_1

$$\begin{aligned} 0 &= \mu_1 \left(1 - 4\mu_1^2 \frac{1}{(1-4\mu_1^2)^2} - \frac{1}{(1-4\mu_1^2)^2}\right) \\ &= \frac{\mu_1(1-4\mu_1^2)^2 - 4\mu_1^3 - \mu_1}{(1-4\mu_1^2)^2} = 0 \iff \\ 0 &= 16\mu_1^5 - 4\mu_1^3 - 8\mu_1^3 \\ &= 16\mu_1^5 - 12\mu_1^3 \end{aligned}$$

This expression is zero if $\mu_1 = 0$ or if $16\mu_1^2 - 12 = 0$, that is if $\mu_1 = \pm\frac{\sqrt{3}}{2}$. The negative solution has to be dropped because the multiplier has to be positive. The positive solution is not valid because it does not satisfy the constrain $g_2 \geq 0$. Therefore there are only one solution in this case,

$$P_1 = \{x = 0, y = 1, \mu_1 = 0, \mu_2 = 0\}$$

- $\mu_1 = \mu_2 = 0$. In this case one retrieves easily solution P_1.
- $\mu_1 \neq 0$ and $1 - x^2 - y^2 = 0$. From the equation of the constraint one has $y = \pm\sqrt{1 - x^2}$, so it is better to split the two cases.
  - $\mu_1 \neq 0$ and $y = +\sqrt{1 - x^2}$. From the equation of the second multiplier one has

$$\mu_2 \left(\sqrt{1 - x^2} - x - \frac{1}{2}\right) = 0 \iff \mu_2 = 0 \quad \text{or} \quad \sqrt{1 - x^2} - x - \frac{1}{2} = 0$$

    * $\mu_2 = 0$, then $x = -2\mu_1\sqrt{1 - x^2}$ hence

$$x^2 = 4\mu_1^2(1 - x^2) \implies x^2(1 + 4\mu_1^2) = 4\mu_1^2 \implies x^2 = \frac{4\mu_1^2}{1 + 4\mu_1^2}.$$

    Thus there are other two cases, i.e. the two square roots for x.

      · When $x = +\sqrt{\frac{4\mu_1^2}{1 + 4\mu_1^2}}$. This gives a single equation for μ_1 which is derived from the equation $\frac{\partial \mathcal{L}}{\partial x}$,

$$\begin{aligned}
0 = \frac{\partial \mathcal{L}}{\partial x} &= \sqrt{1 - \frac{4\mu_1^2}{1+4\mu_1^2}} - 1 + 2\mu_1\sqrt{\frac{4\mu_1^2}{1+4\mu_1^2}} \\
&= \sqrt{\frac{1+4\mu_1^2-4\mu_1^2}{1+4\mu_1^2}} - 1 + 2\mu_1\sqrt{\frac{4\mu_1^2}{1+4\mu_1^2}} \\
&= \frac{1}{\sqrt{1+4\mu_1^2}} - 1 + 2\mu_1\frac{2\mu_1}{\sqrt{1+4\mu_1^2}} = 0 \iff \\
& 1 - \sqrt{1+4\mu_1^2} + 4\mu_1^2 = 0 \iff \\
& 1+4\mu_1^2 = 1 + 8\mu_1^2 + 16\mu_1^4.
\end{aligned}$$

That is when $16\mu_1^4 + 4\mu_1^2 = 0$: $\mu_1 = 0$ is absurd, for hypothesis is $\mu_1 \neq 0$; it remains $16\mu_1^2 = -4$ which has no real solution. So in this case there is no solution.

· When $x = -\sqrt{\frac{4\mu_1^2}{1+4\mu_1^2}}$. This gives a single equation for μ_1 which is derived from the equation $\frac{\partial \mathcal{L}}{\partial x}$,

$$\begin{aligned}
0 = \frac{\partial \mathcal{L}}{\partial x} &= \sqrt{1 - \frac{4\mu_1^2}{1+4\mu_1^2}} - 1 - 2\mu_1\sqrt{\frac{4\mu_1^2}{1+4\mu_1^2}} \\
&= \frac{1}{\sqrt{1+4\mu_1^2}} - 1 - 2\mu_1\frac{2\mu_1}{\sqrt{1+4\mu_1^2}} = 0 \iff \\
& 1 - \sqrt{1+4\mu_1^2} - 4\mu_1^2 = 0 \iff \\
& 1+4\mu_1^2 = 1 - 8\mu_1^2 + 16\mu_1^4.
\end{aligned}$$

That is when $16\mu_1^4 - 12\mu_1^2 = 0$: $\mu_1 = 0$ is absurd, for hypothesis is $\mu_1 \neq 0$; it remains $16\mu_1^2 = 12$ which has solution $\mu_1 = \pm\sqrt{\frac{12}{16}} = \pm\frac{\sqrt{3}}{2}$. The negative solution is not acceptable because the multiplier has to be positive; the solution $\mu_1 = \frac{\sqrt{3}}{2}$ implies $x = -\sqrt{\frac{4\cdot 3/4}{1+4\cdot 3/4}} = -\frac{\sqrt{3}}{2}$, and $y = +\sqrt{1-x^2} = \frac{1}{2}$. But this solution does not satisfy the first equation of the non linear system, namely

$$0 = \frac{\partial \mathcal{L}}{\partial x} = y - 1 + 2\mu_1 x + \mu_2 = \frac{1}{2} - 1 + 2\,\frac{\sqrt{3}}{2}\left(-\frac{\sqrt{3}}{2}\right) \neq 0.$$

So this case has no solution.

* $\mu_2 \neq 0$ but $\sqrt{1-x^2} - x - \dfrac{1}{2} = 0$. There is a single expression for x that gives

$$0 = \sqrt{1-x^2} - x - \frac{1}{2} \implies \sqrt{1-x^2} = x + \frac{1}{2}$$

and removing the square root it remains $2x^2 + x - \frac{3}{4} = 0$ which has solution $x = -\frac{1}{4} \pm \frac{1}{4}\sqrt{7}$. Therefore one obtains y in the two cases:

$$y = +\sqrt{1-x^2} \implies y = \frac{1}{4} + \frac{1}{4}\sqrt{7} \text{ and } y = -\frac{1}{4} + \frac{1}{4}\sqrt{7}.$$

Substituting these values in the non linear system gives a reduced system

$$\begin{aligned} x + 2\mu_1 y - \mu_2 &= 0 \\ y - 1 + 2\mu_1 x + \mu_2 &= 0. \end{aligned}$$

Summing the two equation and substituting $x = -\frac{1}{4} + \frac{1}{4}\sqrt{7}$ and the corresponding $y = \frac{1}{4} + \frac{1}{4}\sqrt{7}$ yields

$$\begin{aligned} &\frac{1}{2}\sqrt{7} - 1 + \frac{1}{2}\mu_1 + \frac{1}{2}\sqrt{7}\mu_1 - \frac{1}{2}\mu_1 + \frac{1}{2}\sqrt{7}\mu_1 = 0 \iff \\ &\frac{1}{2}\sqrt{7} - 1 + \sqrt{7}\mu_1 = 0 \implies \mu_1 = \frac{\sqrt{7}}{7} - \frac{1}{2} \approx -0.12 < 0. \end{aligned}$$

Thus this solution in not acceptable. Checking the second solution for x and y, i.e summing the two equation and substituting $x = -\frac{1}{4} - \frac{1}{4}\sqrt{7}$ and the corresponding $y = -\frac{1}{4} + \frac{1}{4}\sqrt{7}$ yields

$$\begin{aligned} &-\frac{1}{2} - 1 - \frac{1}{2}\mu_1 + \frac{1}{2}\sqrt{7}\mu_1 - \frac{1}{2}\mu_1 - \frac{1}{2}\sqrt{7}\mu_1 = 0 \iff \\ &\mu_1 = -\frac{3}{2} < 0. \end{aligned}$$

So even this solution is not valid.

– $\mu_1 \neq 0$ and $y = -\sqrt{1-x^2}$. From the equation $\mu_2 g_2 = 0$

$$\mu_2\left(-\sqrt{1-x^2} - x - \frac{1}{2}\right) = 0 \iff \mu_2 = 0 \text{ or } -\sqrt{1-x^2} - x - \frac{1}{2} = 0.$$

This leads to the two following different cases.

* $\mu_2 = 0$. In this case the first equation of the non linear system becomes

$$x - 2\mu_1\sqrt{1-x^2} = 0 \implies x^2 = \frac{4\mu_1^2}{1+4\mu_1^2} \implies x = \pm\frac{2\mu_1}{\sqrt{1+4\mu_1^2}}$$

· $x = +\dfrac{2\mu_1}{\sqrt{1+4\mu_1^2}}$. In this case the second equation of the system becomes

$$
\begin{aligned}
&-\sqrt{1-\frac{4\mu_1^2}{1+4\mu_1^2}} - 1 + 2\mu_1\frac{2\mu_1}{\sqrt{1+4\mu_1^2}} = 0 \iff \\
&-1-\sqrt{1+4\mu_1^2}+4\mu_1^2 = 0 \iff 1-8\mu_1^2+16\mu_1^4 = 1+4\mu_1^2 \\
&\implies 16\mu_1^4 - 4\mu_1^2 = 0 \iff \mu_1 = 0, \pm\frac{1}{2}.
\end{aligned}
$$

The solution $\mu_1 = 0$ is absurd by hypothesis, $\mu_1 = -\frac{1}{2}$ is not valid because it is negative, it remains to check if $\mu_1 = \frac{1}{2}$ is acceptable. This implies $\mu_2 = 0$ and

$$
x = \frac{\sqrt{2}}{2} \implies y = -\frac{\sqrt{2}}{2}
$$

but these values do not satisfy the first equation of the system, in facts

$$
-\frac{\sqrt{2}}{2} - 1 + 2\cdot\frac{1}{2}\frac{\sqrt{2}}{2} \neq 0.
$$

So no one of these solutions is valid. Now check the case for the negative values of the root for x.

· $x = -\dfrac{2\mu_1}{\sqrt{1+4\mu_1^2}}$. In this case the second equation of the system becomes

$$
\begin{aligned}
&-\sqrt{1-\frac{4\mu_1^2}{1+4\mu_1^2}} - 1 - 2\mu_1\frac{2\mu_1}{\sqrt{1+4\mu_1^2}} = 0 \iff \\
&-1-\sqrt{1+4\mu_1^2}-4\mu_1^2 = 0 \iff 1+8\mu_1^2+16\mu_1^4 = 1+4\mu_1^2 \\
&\implies 16\mu_1^4 + 4\mu_1^2 = 0 \iff \mu_1 = 0, \pm\boldsymbol{i}\frac{1}{2}.
\end{aligned}
$$

So no solution is acceptable.

∗ $\mu_2 \neq 0$ and $-\sqrt{1-x^2} - x - \dfrac{1}{2} = 0$. This is single equation for x that can be solved squaring the root:

$$
-\sqrt{1-x^2} - x - \frac{1}{2} = 0 \iff x^2 + x + \frac{1}{4} = 1 - x^2 \implies 2x^2 + x - \frac{3}{4}
$$

This case is the same of the case $\mu_2 \neq 0$, $y = +\sqrt{1-x^2}$ and $\sqrt{1-x^2} - x - \dfrac{1}{2} = 0$, so the same conclusion holds: there is no solution.

- $\mu_2 \neq 0$ and $y - x - \frac{1}{2} = 0$. One obtains $y = x + \frac{1}{2}$ and from the equation $\mu_1 g_1 = 0$

$$\mu_1 \left(1 - x^2 - \left(x + \frac{1}{2}\right)^2\right) = \mu_1 \left(2x^2 + x - \frac{3}{4}\right) = 0$$

  There are two cases, $\mu_1 = 0$ and $2x^2 + x - \dfrac{3}{4} = 0$.

  - $\mu_1 = 0$. From the second equation of the system one has $\mu_2 = x$, and from the first equation of the system

$$y - 1 + \mu_2 = x + \frac{1}{2} - 1 + \mu_2 = 0 \implies \mu_2 = \frac{1}{4}.$$

    From that values one has $x = \frac{1}{4}$ and $y = \frac{1}{4} + \frac{1}{2} = \frac{3}{4}$. Further more one can check that this solution satisfy the non linear system and is therefore a valid candidate,

$$P_2 = \left\{x = \frac{1}{4}, y = \frac{3}{4}, \mu_1 = 0, \mu_2 = \frac{1}{4}\right\}.$$

  - $2x^2 + x - \dfrac{3}{4} = 0$. The solution of this quadratic are $x = -\frac{1}{4} \pm \frac{1}{4}\sqrt{7}$ and so $y = \pm\frac{1}{4} + \frac{1}{4}\sqrt{7}$ and $\mu_2 = x$. But these solution do not satisfy the first equation of the system, giving respectively

$$\frac{1}{4} + \frac{1}{4}\sqrt{7} - 1 - \frac{1}{4} + \frac{1}{4}\sqrt{7} \neq 0$$
$$\frac{1}{4} - \frac{1}{4}\sqrt{7} - 1 - \frac{1}{4} - \frac{1}{4}\sqrt{7} \neq 0.$$

In conclusion there are only two candidates to be minima,

$$P_1 = \{x = 0, y = 1, \mu_1 = 0, \mu_2 = 0\}$$
$$P_2 = \left\{x = \frac{1}{4}, y = \frac{3}{4}, \mu_1 = 0, \mu_2 = \frac{1}{4}\right\}.$$

To check if they are maxima, minima or saddle points one has to see if the projected Hessian of the Lagrangian is SPD etc. The Hessian of $\mathcal{L}$ with respect to x, y is

$$\nabla^2_{(x,y)}\mathcal{L} = \nabla^2_{(x,y)}\mathcal{L}(P_1) = \nabla^2_{(x,y)}\mathcal{L}(P_2) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The gradient of the constraints is

$$\nabla G(x, y) = \begin{pmatrix} -2x & -2y \\ -1 & 1 \end{pmatrix}.$$

With KKT conditions one has to project the Hessian of $\mathcal{L}$ only with respect to the active constraints, i.e. those for which $g_i(P_j) = 0$, in this case, $g_1(P_1) = 1 - 0 - 1 = 0$, $g_2(P_1) =$

$1 - 0 - 1/2 \neq 0$ and $g_1(P_2) = 1 - 1/16 - 9/16 \neq 0$, $g_2(P_2) = 3/4 - 1/4 - 1/2 = 0$, so for P_1 is active g_1, for P_2 is active g_2.

$$\nabla g_1(P_1) = \begin{pmatrix} -2x & -2y \end{pmatrix}\Big|_{P_1} = \begin{pmatrix} 0 & -2 \end{pmatrix} \qquad \nabla g_2(P_2) = \begin{pmatrix} -1 & 1 \end{pmatrix}\Big|_{P_2} = \begin{pmatrix} -1 & 1 \end{pmatrix}.$$

A vector in the kernel of $\nabla g_1(P_1)$ is $w_1 = (1,0)^T$, a vector in the kernel of $\nabla g_2(P_2)$ is $w_2 = (1,1)^T$. Hence the projection of the Hessian becomes in the two cases

$$\begin{aligned} w_1^T \nabla^2_{(x,y)} \mathcal{L}(P_1) w_1 &= \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0, \\ w_2^T \nabla^2_{(x,y)} \mathcal{L}(P_2) w_2 &= \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 2. \end{aligned}$$

So P_2 is a minimum point, but nothing can be concluded for P_1.

2 Exercise 2

Solve the following constrained minimization

$$f(x,y,z) = z + xy$$

subject to

$$x^2 + y^2 \leq 1 \qquad\qquad x \leq y + z.$$

2.1 Solution with KKT

First put the constraints in the form of the theorem, i.e. $g_i(x,y,z) \geq 0$, this gives

$$g_1(x,y,z) = 1 - x^2 - y^2 \geq 0 \qquad\qquad g_2(x,y,z) = y + z - x \geq 0.$$

The next step is to build the Lagrangian,

$$\begin{aligned} \mathcal{L}(x,y,\mu_1,\mu_2) &= f(x,y,z) - \mu_1 g_1(x,y,z) - \mu_2 g_2(x,y,z) \\ &= z + xy - \mu_1(1 - x^2 - y^2) - \mu_2\left(y + z - x\right). \end{aligned}$$

So using the (first order) KKT conditions, the associated non linear system is

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial x} &= y + 2\mu_1 x + \mu_2 = 0 \\ \frac{\partial \mathcal{L}}{\partial y} &= x + 2\mu_1 y - \mu_2 = 0 \\ \frac{\partial \mathcal{L}}{\partial z} &= 1 - \mu_2 \\ \mu_1 g_1 &= \mu_1\left(1 - x^2 - y^2\right) = 0 \\ \mu_2 g_2 &= \mu_2\left(y + z - x\right) = 0 \end{aligned}$$

with the conditions

$$\mu_1 \geq 0 \qquad \mu_2 \geq 0 \qquad g_1 \geq 0 \qquad g_2 \geq 0.$$

To solve the non linear system, it is convenient to divide it in the various cases.

- $\mu_1 = 0$. From $\frac{\partial \mathcal{L}}{\partial z}$ one has $\mu_2 = 1$. In this way the general system simplifies to

$$\begin{aligned} &y + z - x = 0 \\ &x - 1 = 0 \\ &y + 1 = 0. \end{aligned}$$

  The solution is trivially $x = 1$, $y = -1$, and from the first equation $-1 + z - 1 = 0 \implies z = 2$. One can check that this is not a valid candidate, because it does not satisfy the constraint $g_1 \geq 0$.

- $\mu_2 = 0$. From $\frac{\partial \mathcal{L}}{\partial z} = 1 \neq 0$, this is absurd, so there is no solution.
- $\mu_1 = \mu_2 = 0$. From $\frac{\partial \mathcal{L}}{\partial z} = 1 \neq 0$, this is absurd, so there is no solution.
- $\mu_1 \neq 0$ and $1 - x^2 - y^2 = 0$. Here there are two subcases:
  - $y = +\sqrt{1 - x^2}$. From the equation $\mu_2 g_2 = 0$ there are two cases,
    * $\mu_2 = 0$. This implies as before $\frac{\partial \mathcal{L}}{\partial z} = 1 \neq 0$, so no solution.
    * $y + z - x = 0$. The resulting system becomes

$$\begin{aligned} &y + z - x = 0 \\ &y = \sqrt{1 - x^2} \\ &x + 2\mu_1\sqrt{1 - x^2} - 1 = 0 \\ &\sqrt{1 - x^2} + 2\mu_1 x + 1 = 0. \end{aligned}$$

      From the last equation one has

$$2\mu_1 x = -1 - \sqrt{1 - x^2} \implies \mu_1 = \frac{-1 - \sqrt{1 - x^2}}{2x} \text{ for } x \neq 0.$$

      Putting this expression in the third equation gives,

$$\begin{aligned} 0 &= x + \frac{-1 - \sqrt{1 - x^2}}{x}\sqrt{1 - x^2} - 1 \\ &= x - \frac{\sqrt{1 - x^2}}{x} - \frac{1 - x^2}{x} - 1 \\ &= \frac{x^2 - \sqrt{1 - x^2} - 1 + x^2 - x}{x} \\ &= \frac{2x^2 - x - 1 - \sqrt{1 - x^2}}{x} \end{aligned}$$

Removing the square root yields

$$4x^4 + x^2 + 1 - 4x^3 - 4x^2 + 2x - 1 + x^2 = 0$$
$$4x^4 - 4x^3 - 2x^2 + 2x = 0$$

This equation has two trivial roots, $x = 0$ and $x = 1$, the other two can be obtained from the reduction of the quartic to a quadratic, and are $x = \pm\frac{\sqrt{2}}{2}$. Now the solution $x = 0$ has to be discarded because of the discussion for μ_1, the solution $x = 1$ gives $\mu_1 = -\frac{1}{2}$ which is not valid, the solution $x = \frac{\sqrt{2}}{2}$ gives $\mu_1 = \frac{-1-\sqrt{1/2}}{\sqrt{2}} = -\frac{1}{2} - \frac{\sqrt{2}}{2} < 0$ and is not valid, finally $x = -\frac{\sqrt{2}}{2}$ gives $\mu_1 = \frac{-1-\sqrt{1/2}}{-\sqrt{2}} = \frac{1}{2} + \frac{\sqrt{2}}{2}$. This $y = y = \frac{\sqrt{2}}{2}$, $z = x - y = -\sqrt{2}$. One can verify that this solution satisfy the two constraints. In conclusion there is only one valid candidate in this case,

$$P = \left\{ x = -\frac{\sqrt{2}}{2}, y = \frac{\sqrt{2}}{2}, z = -\sqrt{2}, \mu_1 = \frac{1+\sqrt{2}}{2}, \mu_2 = 1 \right\}.$$

– $y = -\sqrt{1 - x^2}$. As in the previous case there are two possibilities:
  - $\mu_2 = 0$. In this case the choose $\mu_2 = 0$ produces the same absurd as before.
  - $y + z - x = 0$. The simplified system becomes

$$y + z - x = 0$$
$$y = -\sqrt{1 - x^2}$$
$$x + 2\mu_1\sqrt{1 - x^2} - 1 = 0$$
$$-\sqrt{1 - x^2} + 2\mu_1 x + 1 = 0.$$

From the last equation one has

$$2\mu_1 y = 1 - x \implies \mu_1 = -\frac{1 - x}{2\sqrt{1 - x^2}} \text{ for } x \neq \pm 1.$$

Putting this expression in the third equation gives,

$$\begin{aligned} 0 &= -\sqrt{1 - x^2} - \frac{1 - x}{\sqrt{1 - x^2}} x + 1 \\ &= -(1 - x^2) - x + x^2 + \sqrt{1 - x^2} \\ &= 4x^4 + x^2 + 1 - 4x^3 - 4x^2 + 2x - 1 + x^2 \\ &= 4x^4 - 4x^3 - 2x^2 + 2x \end{aligned}$$

This is the quartic of the previous case, this time $x = 1$ is not a valid solution, $x = \frac{\sqrt{2}}{2}$ gives $\mu_1 = -\frac{1-\sqrt{2}/2}{2\sqrt{1/2}} = \frac{1}{2} - \frac{\sqrt{2}}{2} < 0$ and is not valid, $x = -\frac{\sqrt{2}}{2}$ gives $\mu_1 = -\frac{1}{2} - \frac{\sqrt{2}}{2} < 0$ is not valid. So in this case there are no solutions.

- $\mu_2 \neq 0$ and $y+z-x=0$. Here there are two subcases
  - $\mu_1 = 0$. The simplified system becomes

$$\begin{aligned} y+z-x&=0\\ y+1&=0\\ x-1&=0 \end{aligned}$$

    so there is the trivial solution $x = 1$, $y = -1$, $z = 2$ but does not satisfy the constraints.
  - $1-x^2-y^2=0$. There the two subcases $y = \pm\sqrt{1-x^2}$, but are identical as those done before, so they give the same result.

In conclusion there is only one valid candidate, namely

$$P = \left\{ x = -\frac{\sqrt{2}}{2}, y = \frac{\sqrt{2}}{2}, z = -\sqrt{2}, \mu_1 = \frac{1+\sqrt{2}}{2}, \mu_2 = 1 \right\}.$$

The Hessian of the Lagrangian is

$$\nabla^2_{(x,y,z)}\mathcal{L} = \begin{pmatrix} 2\mu_1 & 1 & 0\\ 1 & 2\mu_1 & 0\\ 0 & 0 & 0 \end{pmatrix}.$$

The gradient of the constraints is

$$\nabla G = \begin{pmatrix} -2x & -2y & 0\\ -1 & -1 & 1 \end{pmatrix}.$$

Now it is necessary to check if the constraints are active, i.e. if $g_1(P) = 0$ or $g_2(P) = 0$. Thus

$$\begin{aligned} g_1(P) &= 1 - \frac{1}{2} - \frac{1}{2} = 0\\ g_2(P) &= \frac{\sqrt{2}}{2} - \sqrt{2} + \frac{\sqrt{2}}{2} = 0 \end{aligned}$$

So both constraints are active and the whole gradient has to be considered

$$\nabla G(P) = \begin{pmatrix} \sqrt{2} & -\sqrt{2} & 0\\ -1 & -1 & 1 \end{pmatrix}.$$

To find a vector $w = (\alpha, \beta, \gamma)^T$ in the kernel of $\nabla G(P)$ one can solve this linear system

$$0 = \nabla G(P)w = \begin{pmatrix} \sqrt{2} & -\sqrt{2} & 0\\ -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} \alpha\\ \beta\\ \gamma \end{pmatrix} = \begin{pmatrix} \sqrt{2}\alpha - \sqrt{2}\beta\\ -\alpha + \beta + \gamma \end{pmatrix}$$

A possible solution is $w = (1, 1, 0)^T$. The projection of the Hessian in this kernel is thus

$$w^T \nabla^2_{(x,y,z)} \mathcal{L}(P) w = \begin{pmatrix} 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1+\sqrt{2} & 1 & 0 \\ 0 & 1+\sqrt{2} & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} =$$

$$\begin{pmatrix} 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1+\sqrt{2}+1 \\ 1+\sqrt{2}+1 \\ 0 \end{pmatrix} = 1+\sqrt{2}+1+1+\sqrt{2}+1 = 4+2\sqrt{2} > 0.$$

Hence P is a minimum point.