Exercitation 8

Numerical Methods for Dynamical Systems and Control

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1 Exercise 1

Compute the exponential of the following matrix.

$$\boldsymbol{A} = \begin{pmatrix} 17 & -20 \\ 12 & -14 \end{pmatrix}$$

1.1 Solution with Eigenvectors

If $\boldsymbol{A}$ has a full set of eigenvectors and eigenvalues, then $\boldsymbol{A}$ can be diagonalized in $\boldsymbol{A} = \boldsymbol{P}\boldsymbol{D}\boldsymbol{P}^{-1}$, where $\boldsymbol{D}$ is the matrix whose diagonal are the eigenvalues of $\boldsymbol{A}$, and $\boldsymbol{P}$ is the matrix of the eigenvectors.
The characteristic polynomial of matrix A is

$$\begin{aligned} \det(\boldsymbol{A} - \lambda \boldsymbol{I}) &= \det \begin{pmatrix} 17-\lambda & -20 \\ 12 & -14-\lambda \end{pmatrix} = (17-\lambda)(-14-\lambda) + 12 \cdot 20 \\ &= -17 \cdot 14 - 17\lambda + 14\lambda + \lambda^2 + 12 \cdot 20 \\ &= -238 - 3\lambda + \lambda^2 + 240 \\ &= \lambda^2 - 3\lambda + 2. \end{aligned}$$

The roots of the characteristic polynomial are $\lambda = 1, 2$. There are two distinct eigenvalues, so the matrix is diagonalizable. To compute the exponential one needs a basis of eigenvectors, they can be calculated as follows.

$$\boldsymbol{A}\boldsymbol{v} = \lambda \boldsymbol{v} \implies \begin{pmatrix} 17 & -20 \\ 12 & -14 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \lambda \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

This matrix product ends up in a linear system, with a non zero solution, for example summing the two linear equations.

$$\boldsymbol{A}\boldsymbol{v} - \lambda \boldsymbol{v} = \boldsymbol{0} \implies \begin{pmatrix} 17\alpha - 20\beta - \lambda\alpha \\ 12\alpha - 14\beta - \lambda\beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For the first eigenvalue $\lambda = 1$, the sum of this two equations leads to

$$29\alpha - 34\beta - \alpha - \beta = 0 \implies \alpha = \frac{35}{28}\beta = \frac{5}{4}\beta.$$

For the second eigenvalue $\lambda = 2$, the sum of the two equations leads to

$$29\alpha - 34\beta - 2\alpha - 2\beta = 0 \implies \alpha = \frac{36}{27}\beta = \frac{4}{3}\beta.$$

To obtain two eigenvectors $\boldsymbol{v}_1, \boldsymbol{v}_2$, one can substitute an arbitrary non zero value for the two betas, e.g. $\beta = 1$. Hence one has $\boldsymbol{v}_1 = (\frac{5}{4}, 1)^T$ and $\boldsymbol{v}_2 = (\frac{4}{3}, 1)^T$. Therefore the matrix $\boldsymbol{P}$ of eigenvectors is

$$\boldsymbol{P} = \begin{pmatrix} \frac{5}{4} & 1 \\ \frac{4}{3} & 1 \end{pmatrix} \implies \boldsymbol{P}^{-1} = -12 \begin{pmatrix} 1 & -\frac{4}{3} \\ -1 & \frac{5}{4} \end{pmatrix} = \begin{pmatrix} -12 & 16 \\ 12 & -15 \end{pmatrix}.$$

The matrix exponential is defined via the Taylor's series

$$e^{\boldsymbol{A}} = \sum_{k=0}^{\infty} \frac{\boldsymbol{A}^k}{k!} = \sum_{k=0}^{\infty} \frac{\boldsymbol{P}\boldsymbol{D}^k\boldsymbol{P}^{-1}}{k!} = \boldsymbol{P}\left(\sum_{k=0}^{\infty} \frac{\boldsymbol{D}^k}{k!}\right)\boldsymbol{P}^{-1} = \boldsymbol{P}e^{\boldsymbol{D}}\boldsymbol{P}^{-1}.$$

In facts, $\boldsymbol{A}^k = \boldsymbol{P}\boldsymbol{D}\boldsymbol{P}^{-1}\boldsymbol{P}\boldsymbol{D}\boldsymbol{P}^{-1} \cdots \boldsymbol{P}\boldsymbol{D}\boldsymbol{P}^{-1} = \boldsymbol{P}\boldsymbol{D}^k\boldsymbol{P}^{-1}$, and the exponential of a diagonal matrix is the exponential of the diagonal elements, i.e. the eigenvalues. In conclusion one has

$$\begin{aligned} e^{\boldsymbol{A}} = \boldsymbol{P}e^{\boldsymbol{D}}\boldsymbol{P}^{-1} &= \begin{pmatrix} \frac{5}{4} & 1 \\ \frac{4}{3} & 1 \end{pmatrix} \begin{pmatrix} e & 0 \\ 0 & e^2 \end{pmatrix} \begin{pmatrix} -12 & 16 \\ 12 & -15 \end{pmatrix} = \begin{pmatrix} \frac{5}{4} & 1 \\ \frac{4}{3} & 1 \end{pmatrix} \begin{pmatrix} -12e & 16e \\ 12e^2 & -15e^2 \end{pmatrix} \\ &= \begin{pmatrix} -15e + 16e^2 & 20e - 20e^2 \\ 12e^2 - 12e & 15e - 15e^2 \end{pmatrix} \end{aligned}$$

1.2 Solution with the Cayley-Hamilton theorem

From the Taylor's expansion of the matrix exponential, one can consider $e^{\boldsymbol{A}}$ as a polynomial $p(x)$ evaluated in $x = \boldsymbol{A}$:

$$e^{\boldsymbol{A}} = p(x)\big|_{x=\boldsymbol{A}} = \sum_{k=0}^{\infty} \frac{x^k}{k!}\bigg|_{x=\boldsymbol{A}}.$$

From the theorem of Cayley-Hamilton, the characteristic polynomial $\Delta(x)$ of matrix $\boldsymbol{A}$, that is $\Delta(x) = \det(\boldsymbol{A} - x\boldsymbol{I})$, has $x = \boldsymbol{A}$ as a root. Starting from this result one can perform long division of $p(x)$ divided by $\Delta(x)$, yielding

$$p(x) = \Delta(x) \cdot q(x) + r(x)$$

where $q(x)$ is the quotient polynomial and $r(x)$ is the remainder of the division. Using the fact that the degree of $r(x)$ is strictly less than the degree of $\Delta(x)$, i.e. $\partial r(x) < \partial\Delta(x) = n$. Now evaluating $p(x)$ in $x = \boldsymbol{A}$, one has

$$p(x)\big|_{x=\boldsymbol{A}} = p(\boldsymbol{A}) = e^{\boldsymbol{A}} = \Delta(\boldsymbol{A}) \cdot q(\boldsymbol{A}) + r(\boldsymbol{A}) = r(\boldsymbol{A}).$$

Therefore the matrix exponential can be computed as a polynomial in $x = \boldsymbol{A}$ of finite degree. In this exercise $n = 2$ so $p(x) = a_0 + a_1 x$ is a polynomial of degree 1. To find the necessary relation for the coefficient of $p(x)$ one can use the property that if there are n distinct eigenvalues, then $p(\lambda_i) = e^{\lambda_i}$. In this exercise one has

$$\begin{aligned} p(\lambda_1) = p(1) = e^1 &= a_0 + a_1 \\ p(\lambda_2) = p(2) = e^2 &= a_0 + 2a_1 \end{aligned}$$

thus the associated linear system is

$$\begin{aligned} e^1 &= a_0 + a_1 \\ e^2 &= a_0 + 2a_1. \end{aligned}$$

The solution of the system is $a_0 = 2e - e^2$ and $a_1 = e^2 - e$. Hence the matrix exponential becomes

$$\begin{aligned} e^{\boldsymbol{A}} &= p(\boldsymbol{A}) = a_0\boldsymbol{I} + a_1\boldsymbol{A} \\ &= (2e - e^2)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + (e^2 - e)\begin{pmatrix} 17 & -20 \\ 12 & -14 \end{pmatrix} \\ &= \begin{pmatrix} 2e - e^2 + 17e^2 - 17e & -20e^2 + 20e \\ 12e^2 - 12e & 2e - e^2 + 14e - 14e^2 \end{pmatrix} \\ &= \begin{pmatrix} -15e + 16e^2 & 20e - 20e^2 \\ 12e^2 - 12e & 15e - 15e^2 \end{pmatrix}. \end{aligned}$$

2 Exercise 2

Solve the following variational problem.

$$\max \int_0^1 2x' - x^2 + 2x \, dt \qquad x(0) = 1,\ x(1) = 0.$$

2.1 Solution

The first thing to do is to restate the problem as a problem of minimum, this can be done changing the sign of the integrand. Then the problem can be solved easily by using the Euler-Lagrange equation,

$$\frac{d}{dt}\frac{\partial f}{\partial x'} = \frac{\partial f}{\partial x}. \tag{1}$$

One has $f(t,x,x') = -(2x' - x^2 + 2x)$ so $\frac{\partial f}{\partial x} = 2x - 2$ and $\frac{\partial f}{\partial x'} = -2$, hence the Euler-Lagrange equation becomes

$$2x - 2 = 0 \implies x(t) = 1.$$

3 Exercise 3

Solve the following variational problem.

$$\min \int_0^1 (x')^2 + 10tx\, dt \qquad x(0) = 1,\; x(1) = 2.$$

3.1 Solution

The problem can be solved easily by using the Euler-Lagrange equation, as in (1). One has $f(t,x,x') = (x')^2 + 10tx$ so $\frac{\partial f}{\partial x} = 10t$ and $\frac{\partial f}{\partial x'} = 2x'$, thus $\frac{d}{dt}\frac{\partial f}{\partial x'} = \frac{d}{dt}2x' = 2x''$, hence the Euler-Lagrange equation becomes

$$2x'' = 10t \implies x'' = 5t \implies x' = \frac{5}{2}t^2 + c_1 \implies x = \frac{5}{6}t^3 + c_1 t + c_2$$

The constants c_1 and c_2 can be obtained using the initial conditions,

$$\begin{aligned} x(0) &= c_2 = 1 \\ x(1) &= \frac{5}{6} + c_1 + c_2 = 2, \end{aligned}$$

that is $c_1 = 1/6$ and $c_2 = 1$.
In conclusion the minimizing $x(t)$ is

$$\boxed{x(t) = \frac{5}{6}t^3 + \frac{1}{6}t + 1}.$$

4 Exercise 4

Solve the following variational problem.

$$\min \int_0^{\sqrt{\frac{3}{2}}} 2x^2 + 3(x')^2\, dt \qquad x(0) = 1,\; x\left(\sqrt{\frac{3}{2}}\right) = e.$$

4.1 Solution

The problem can be solved easily by using the Euler-Lagrange equation, as in (1). One has $f(t,x,x') = 2x^2 + 3(x')^2$ so $\frac{\partial f}{\partial x} = 4x$ and $\frac{\partial f}{\partial x'} = 6x'$, thus $\frac{d}{dt}\frac{\partial f}{\partial x'} = \frac{d}{dt}6x' = 6x''$, hence the Euler-Lagrange equation becomes

$$6x'' = 4x \implies 6x'' - 4x = 0 \implies 6\lambda^2 - 4 = 0 \implies \lambda = \pm\sqrt{\frac{2}{3}}.$$

Because the differential equation is homogeneous, the solution is only

$$x(t) = c_1 e^{\sqrt{\frac{2}{3}}t} + c_2 e^{-\sqrt{\frac{2}{3}}t}$$

The constants c_1 and c_2 can be obtained using the initial conditions,

$$x(0) = c_1 + c_2 = 1$$

$$x\left(\sqrt{\frac{3}{2}}\right) = e = c_1 e + c_2 e,$$

that is $c_1 = 1$ and $c_2 = 0$.
In conclusion the minimizing $x(t)$ is

$$\boxed{x(t) = e^{\sqrt{\frac{2}{3}}t}}.$$