Exercitation 9

Numerical Methods for Dynamical Systems and Control

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1 Exercise 1

Solve the following system of finite difference equations.

$$x_{k+2} = 2x_{k+1} - x_k$$
$$y_{k+1} = 1 + x_k$$

with initial conditions $x_0 = 0$, $x_1 = 1$, $y_0 = 1$.

1.1 Solution with Z-transform

The transformed system becomes

$$z^2 X - z = 2zX - X$$
$$zY - z = \frac{z}{z-1} + X.$$

Reducing the system, from the first equation

$$X = \frac{z}{z^2 - 2z + 1} = \frac{z}{(z-1)^2}$$

Applying the inverse Z-transform yields

$$\mathcal{Z}^{-1}\left\{\frac{z}{(z-1)^2}\right\} = k.$$

In conclusion the solution for x_k is

$$\boxed{x_k = k}.$$

The solution for y_k can be obtained from the second equation of the system, which leads (for $k \geq 1$) to

$$y_{k+1} = 1 + x_k \implies y_k = 1 + x_{k-1} = 1 + k - 1 = k.$$

One can notice that this solution does not meet the requirements for $y_0 = 1$ so one way is to adjust y_k in zero with a delta, i.e. $y_k = k + \delta_0(k)$. Of course one can solve the second equation in z for Y and perform the inverse transform, this gives

$$Y = \frac{X}{z} + \frac{1}{z-1} + 1 = \frac{1}{(z-1)^2} + \frac{1}{z-1} + 1$$

and doing the least common multiple

$$Y = \frac{1 + z - 1 + z^2 - 2z + 1}{(z-1)^2} = \frac{z^2 - z + 1}{(z-1)^2} = \frac{Az}{z-1} + \frac{Bz}{(z-1)^2}.$$

The easiest way to compute the partial fraction reduction is to consider the expression Y/z:

$$\frac{Y}{z} = \frac{z^2 - z + 1}{z(z-1)^2} = \frac{A}{z-1} + \frac{B}{(z-1)^2} + \frac{C}{z}.$$

By direct substitution one finds out $B = 1$ and $C = 1$. To compute coefficient A one can multiply Y/z by z and then push $z \to \infty$,

$$\lim_{z\to\infty} \frac{Yz}{z} = \lim_{z\to\infty} \frac{z^2 - z + 1}{(z-1)^2} = \lim_{z\to\infty} \frac{Az}{z-1} + \frac{Bz}{(z-1)^2} + \frac{Cz}{z}.$$

This implies the linear equation $A + 0 + C = 1$, that is $A = 0$. So the partial fraction decomposition for Y is

$$Y = \frac{z}{(z-1)^2} + \frac{z}{z}.$$

Thus the inverse Z-transform implies

$$\boxed{y_k = k + \delta_0(k)}.$$

2 Exercise 2

Solve this system of differential equations.

$$x'' - y'' = e^t$$

$$x + y = 0$$

with $x(0) = 1$, $x'(0) = 0$, $y(0) = -1$ and $y'(0) = 0$.

2.1 Solution with Laplace transform

The transformed system becomes

$$\begin{aligned} &s^2X - sx(0) - x'(0) - s^2Y + sy(0) + y'(0) = \frac{1}{s-1} \\ &X + Y = 0. \end{aligned}$$

Focusing on the first equation, the substitution of the initial values gives

$$s^2X - 2s - s^2Y = \frac{1}{s-1}$$

Using the second equation gives $Y = -X$ so the first equation reduces to

$$X = \frac{1}{s-1}\frac{1}{2s^2} + \frac{2s}{2s^2} = \frac{1}{2}\frac{1}{s^2(s-1)} + \frac{1}{s}.$$

The partial fraction decomposition of the first term gives

$$X = \frac{1}{2}\left(\frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-1}\right) + \frac{1}{s}.$$

By direct substitution one has $B = -1$ and $C = 1$, multiplying by s and pushing $s \to \infty$

$$\lim_{s\to\infty} \frac{s}{s^2(s-1)} = \lim_{s\to\infty} A + \frac{B}{s} + C = 0$$

therefore $A + C = 0$ that is $A = -1$. In conclusion

$$X = \frac{1}{2}\left(-\frac{1}{s} - \frac{1}{s^2} + \frac{1}{s-1}\right) + \frac{1}{s} \implies x(t) = -\frac{1}{2} - \frac{t}{2} - \frac{e^t}{2} + 1.$$

So the required solution is $\boxed{x(t) = \frac{1}{2} - \frac{t}{2} - \frac{e^t}{2}}$ and $y(t) = -x(t)$.

3 Exercise 3

Solve the differential equation

$$y'''(t) = \cos(t) + A, \qquad y(0) = 1,\, y'(0) = 1,\, y''(0) = A$$

and determine A such that $\lim_{t\to\infty} \frac{y(t)}{t^3} = 1$.

3.1 Solution with calculus

The exercise can be done without using particular ODE techniques, in fact the equation can be integrated three times. One has

$$y''(t) = \sin(t) + At + c_1 \implies y''(0) = A = c_1$$

thus $c_1 = A$. Integrating again

$$y'(t) = -\cos(t) + \frac{At^2}{2} + At + c_2 \implies y'(0) = 1 = -1 + c_2$$

hence $c_2 = 2$, and integrating another time,

$$y(t) = -\sin(t) + \frac{At^3}{6} + \frac{At^2}{2} + 2t + c_3 \implies y(0) = 1 = c_3$$

therefore $c_3 = 1$, so the general expression for $y(t)$ is

$$y(t) = -\sin(t) + \frac{At^3}{6} + \frac{At^2}{2} + 2t + 1$$

Now from the condition of the limit one has

$$1 = \lim_{t\to\infty} \frac{y(t)}{t^3} = \lim_{t\to\infty} -\frac{\sin(t) + \frac{At^3}{6} + \frac{At^2}{2} + 2t + 1}{t^3} = \frac{A}{6} \tag{1}$$

thus $A = 6$.

3.2 Solution with Laplace Transform

The only problem is to perform the Laplace transform of a third derivative,

$$\mathcal{L}(y'''(t)) = s^3 Y - s^2 y(0) - sy'(0) - y''(0)$$

so the transformation of the equation yields

$$s^3 Y - s^2 y(0) - sy'(0) - y''(0) = \frac{s}{s^2+1} + \frac{A}{s}.$$

The substitution of the initial conditions gives

$$s^3 Y - s^2 - s - A = \frac{s}{s^2+1} + \frac{A}{s} \implies Y = \frac{A}{s^3} + \frac{1}{s^2} + \frac{1}{s} + \frac{1}{s^2(s^2+1)} + \frac{A}{s^4}$$

The partial fraction decomposition of $1/(s^2(s^2+1))$ is

$$\frac{1}{s^2(s^2+1)} = \frac{\alpha}{s} + \frac{\beta}{s^2} + \frac{\gamma}{s^2+1}$$

and with the usual methods, $\beta = 1$. Then

$$\lim_{s\to\infty} \frac{s}{s^2(s^2+1)} = \alpha = 0 \implies \alpha = 0.$$

Finally,

$$\frac{1}{s^2(s^2+1)} - \frac{\beta}{s^2} - \frac{\alpha}{s} = \frac{\gamma}{s^2+1} \implies \frac{-s^2}{s^2(s^2+1)} = \frac{\gamma}{s^2+1} \implies \gamma = -1.$$

The equation for Y is thus

$$Y = \frac{A}{s^3} + \frac{1}{s^2} + \frac{1}{s} + \frac{1}{s^2} - \frac{1}{s^2+1} + \frac{A}{s^4}$$

and the inversion gives

$$y(t) = \frac{At^2}{2} + t + 1 + t - \sin(t) + \frac{At^3}{6}$$

and one can apply the condition of the limit as in (1).

4 Exercise 4

Write the coefficients of the Fourier series of $f(x) = x\cos(x)$ defined in $(-\pi, \pi)$ and extended periodically in $\mathbb{R}$.

4.1 Solution with real coefficients

The coefficient a_n are all zero because the integral

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos(x)\cos(nx)\,dx = 0$$

is identically zero, in facts the integrand is an odd function integrated on a symmetric interval. It remains to compute the integral for b_n.

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos(x)\sin(nx)\,dx && \text{[Euler formula]}\\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} x\,\frac{e^{\boldsymbol{i}x}+e^{-\boldsymbol{i}x}}{2}\,\frac{e^{\boldsymbol{i}nx}-e^{-\boldsymbol{i}nx}}{2\boldsymbol{i}}\,dx\\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{x}{4\boldsymbol{i}}\left(e^{\boldsymbol{i}x(n+1)} - e^{-\boldsymbol{i}x(n-1)} + e^{\boldsymbol{i}x(n-1)} - e^{-\boldsymbol{i}x(n+1)}\right)\,dx\\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{x}{4\boldsymbol{i}}\left(2\boldsymbol{i}\sin(n+1)x + 2\boldsymbol{i}\sin(n-1)x\right)\,dx && \text{[by parts]}\\
&= \frac{1}{2\pi}\left[x\left(-\frac{1}{n+1}\right)\cos(n+1)x - x\frac{1}{n-1}\cos(n-1)x\right]\Bigg|_{-\pi}^{\pi}\\
&\quad -\frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\left(-\frac{1}{n+1}\right)\cos(n+1)x - \frac{1}{n-1}\cos(n-1)x\right]\,dx\\
&= \frac{1}{2}\left[-\frac{1}{n+1}(-1)^{n+1} - \frac{1}{n-1}(-1)^{n+1} - \frac{1}{n+1}(-1)^{n+1} - \frac{1}{n-1}(-1)^{n+1}\right]\\
&\quad -\frac{2}{2\pi}\left[-\frac{1}{(n+1)^2}\sin(n+1)x - \frac{1}{(n-1)^2}\sin(n-1)x\right]\Bigg|_{0}^{\pi}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\left[\frac{(-1)^n}{n+1} + \frac{(-1)^n}{n-1} + \frac{(-1)^n}{n+1} + \frac{(-1)^n}{n-1}\right] - 0 \\
&= \frac{(-1)^n}{2}\left[\frac{n-1+n+1+n-1+n+1}{n^2-1}\right] \\
&= \frac{(-1)^n}{2}\left[\frac{4n}{n^2-1}\right] \\
&= \frac{(-1)^n 2n}{n^2-1}.
\end{aligned}
$$

In conclusion the required coefficients are

$$
a_n = 0 \qquad\qquad b_n = \frac{(-1)^n 2n}{n^2-1} \qquad n \geq 2.
$$