Exercitation 10

Numerical Methods for Dynamical Systems and Control

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1 Exercise 1

Compute the matrix exponential $e^{\boldsymbol{A}}$ where

$$\boldsymbol{A} = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}.$$

1.1 Solution with Cayley-Hamilton

The first thing to do is to find the eigenvalues, this can be done solving the equation $\det(\boldsymbol{A} - \lambda \boldsymbol{I}) = 0$.

$$\boldsymbol{A} - \lambda\boldsymbol{I} = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2-\lambda & -1 & -1 \\ -1 & 2-\lambda & -1 \\ -1 & -1 & 2-\lambda \end{pmatrix}$$

Its determinant is

$$\begin{aligned} 0 = \det(\boldsymbol{A} - \lambda\boldsymbol{I}) &= (2-\lambda)[(2-\lambda)^2 - 1] + [(\lambda - 2) - 1] - [1 + (2-\lambda)] \\ &= (2-\lambda)^3 - (2-\lambda) + (\lambda - 2 - 1 - 1 - 2 + \lambda) \\ &= 8 - 12\lambda + 6\lambda^2 - \lambda^3 - 2 + \lambda + 2\lambda - 6 \\ &= -\lambda^3 + 6\lambda^2 - 9\lambda \\ &= \lambda(\lambda - 3)^2. \end{aligned}$$

The solution of that equation are $\lambda = 0$ and a double root $\lambda = 3$. Using the Cayley-Hamilton theorem, one has to compute the polynomial $p(x) = a_0 + a_1 x + a_2 x^2$ using the relations $p(\lambda = 0) = e^0$, $p(\lambda = 3) = e^3$ and $p'(\lambda = 3) = e^3$. This gives a non linear system in the coefficients a_i, namely

$$p(0) = a_0 = e^0$$
$$p(3) = a_0 + 3a_1 + 9a_2 = e^3$$
$$p'(3) = a_1 + 6a_2 = e^3.$$

Using the first equation, $a_0 = 1$, substituting this value in the other two equation yields from the third equation, $a_1 = e^3 - 6a_2$. Using this expression in the second equation gives $1 + 3e^3 - 18a_2 + 9a_2 = e^3$ from which

$$\begin{aligned} a_0 &= 1 \\ a_1 = e^3 - 6a_2 = \frac{9e^3 - 6 - 12e^3}{9} &= \frac{-3e^3 - 6}{9} \\ a_2 &= \frac{1 + 2e^3}{9}. \end{aligned}$$

In conclusion the required expression $e^{\boldsymbol{A}} = p(\boldsymbol{A})$, that is

$$p(\boldsymbol{A}) = a_0\boldsymbol{I} + a_1\boldsymbol{A} + a_2\boldsymbol{A}^2 = \boldsymbol{I} - \frac{e^3 + 2}{3}\boldsymbol{A} + \frac{1 + 2e^3}{9}\boldsymbol{A}^2$$

One can compute $\boldsymbol{A}^2$ and obtains

$$\boldsymbol{A}^2 = \begin{pmatrix} 6 & -3 & -3 \\ -3 & 6 & -3 \\ -3 & -3 & 6 \end{pmatrix}.$$

Putting all together

$$\begin{pmatrix} 1/3 + 2/3\,e^3 & -1/3\,e^3 + 1/3 & -1/3\,e^3 + 1/3 \\ -1/3\,e^3 + 1/3 & 1/3 + 2/3\,e^3 & -1/3\,e^3 + 1/3 \\ -1/3\,e^3 + 1/3 & -1/3\,e^3 + 1/3 & 1/3 + 2/3\,e^3 \end{pmatrix}.$$

1.2 Solution with standard linear algebra

From the previous section, the eigenvalues of the matrix are $\lambda = 0$ and a double root $\lambda = 3$. So the algebraic multiplicity of 0 is $a.m.(0) = 1$ and $a.m.(3) = 2$. To find a basis of eigenvectors one need to check if the geometric multiplicity of the eigenvalues is equal to the algebraic multiplicity.

The rank of $\boldsymbol{A} - 0\boldsymbol{I}$ is equal to the rank of $\boldsymbol{A}$ and is 2, in facts the first row is equal to the sum of the other two rows with opposite sign. So the geometric multiplicity of 0 is $g.m.(0) = n - rk(\boldsymbol{A} - \lambda\boldsymbol{I}) = 3 - 2 = 1$ and is equal to the algebraic multiplicity of 0.

The same computation for $\lambda = 3$ gives

$$\boldsymbol{A} - 3\boldsymbol{I} = \begin{pmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{pmatrix}$$

hence this matrix has rank 1 and so $g.m.(3) = 3 - 1 = 2 = a.m.(3)$. So there are enough eigenvectors to construct the matrix $\boldsymbol{P}$ of eigenvectors.
The eigenvector corresponding to 0 is a non zero vector of the kernel of $\boldsymbol{A}$, i.e. a vector $v_0 = (\alpha, \beta, \gamma)^T$ such that $\boldsymbol{A}v_0 = \mathbf{0}$.

$$\boldsymbol{A}v_0 = (2\alpha - \beta - \gamma, -\alpha + 2\beta - \gamma, -\alpha - \beta + 2\gamma)^T = \mathbf{0}$$

One possible solution is $v_0 = (1, 1, 1)^T$.
For $\lambda = 3$ there are two eigenvectors which must satisfy

$$\boldsymbol{A}v = (2\alpha - \beta - \gamma, -\alpha + 2\beta - \gamma, -\alpha - \beta + 2\gamma)^T = (3\alpha, 3\beta, 3\gamma)^T$$

summing the first two relations one has $\alpha + \beta + \gamma = 0$ and two linearly independent solution of this are $v_1 = (1, -1, 0)^T$ and $v_2 = (1, 0, -1)^T$.
The matrix $\boldsymbol{P}$ of eigenvectors is therefore

$$\boldsymbol{P} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \implies \boldsymbol{P}^{-1} = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}$$

In conclusion the exponetial of $\boldsymbol{A}$ is

$$\begin{aligned} \boldsymbol{P}e^{\boldsymbol{D}}\boldsymbol{P}^{-1} &= \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} e^0 & 0 & 0 \\ 0 & e^3 & 0 \\ 0 & 0 & e^3 \end{pmatrix} \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \\ &= \begin{pmatrix} 1/3 + 2/3\, e^3 & -1/3\, e^3 + 1/3 & -1/3\, e^3 + 1/3 \\ -1/3\, e^3 + 1/3 & 1/3 + 2/3\, e^3 & -1/3\, e^3 + 1/3 \\ -1/3\, e^3 + 1/3 & -1/3\, e^3 + 1/3 & 1/3 + 2/3\, e^3 \end{pmatrix}. \end{aligned}$$

2 Exercise 2

Compute the matrix exponential $e^{\boldsymbol{A}}$ where

$$\boldsymbol{A} = \begin{pmatrix} -7 & -5 & 5 & 10 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ -5 & -5 & 5 & 8 \end{pmatrix}.$$

2.1 Solution with linear algebra

The determinant of this matrix can be quite involved if expanded using the first row, but reduces to a single 3×3 determinant if expanded using the Laplace determinant formula applied to the second or third row. Expanding along the third row one has

$$\begin{aligned}
\det(\boldsymbol{A} - \lambda \boldsymbol{I}) &= \det \begin{pmatrix} -7-\lambda & -5 & 5 & 10 \\ 0 & -2-\lambda & 0 & 0 \\ 0 & 0 & -2-\lambda & 0 \\ -5 & -5 & 5 & 8-\lambda \end{pmatrix} \\
&= (-2-\lambda) \det \begin{pmatrix} -7-\lambda & 5 & 10 \\ 0 & -2-\lambda & 0 \\ -5 & 5 & 8-\lambda \end{pmatrix} \\
&= (-2-\lambda)(-2-\lambda) \det \begin{pmatrix} -7-\lambda & 10 \\ -5 & 8-\lambda \end{pmatrix} \\
&= (-2-\lambda)^2[(7-\lambda)(8-\lambda)+50] \\
&= (-2-\lambda)^2[\lambda^2-\lambda-6]
\end{aligned}$$

This equation has roots $\lambda = -2, 3$ with $a.m.(-2) = 3$ and $a.m.(3) = 1$. Now there is to check the geometric multiplicity. For the first eigenvalue:

$$\boldsymbol{A} + 2\boldsymbol{I} = \begin{pmatrix} -5 & -5 & 5 & 10 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -5 & -5 & 5 & 10 \end{pmatrix}$$

and this means that it has rank 1, so $g.m.(-2) = 4 - 1 = 3 = a.m.(-2)$. For the second eigenvalue one has

$$\boldsymbol{A} - 3\boldsymbol{I} = \begin{pmatrix} -10 & -5 & 5 & 10 \\ 0 & -5 & 0 & 0 \\ 0 & 0 & -5 & 0 \\ -5 & -5 & 5 & 5 \end{pmatrix} \implies \begin{pmatrix} 0 & -5 & 5 & 0 \\ 0 & -5 & 0 & 0 \\ 0 & 0 & -5 & 0 \\ -5 & -5 & 5 & 5 \end{pmatrix} \implies \begin{pmatrix} 0 & 0 & -5 & 0 \\ 0 & -5 & 0 & 0 \\ 0 & 0 & -5 & 0 \\ -5 & -5 & 5 & 5 \end{pmatrix}$$

where in the first passage were used the first and fourth rows, in the second passage, the second and the first. It turns out that it has rank 3, so $g.m.(3) = 4-3 = 1 = a.m.(3)$. Therefore there exists a basis of eigenvectors. For $\lambda = -2$ three eigenvectors can be $v_1 = (1, 0, -1, 1)^T$, $v_2 = (0, 1, 1, 0)^T$ and $v_3 = (1, -1, 0, 0)^T$, because all satisfy $(\boldsymbol{A} - 3\boldsymbol{I})v = \boldsymbol{0}$. For $\lambda = 3$ one has that $v_4 = (1, 0, 0, 1)^T$ satisfies $(\boldsymbol{A} + 2\boldsymbol{I})v_4 = \boldsymbol{0}$. Thus the matrix $\boldsymbol{P}$ is

$$\boldsymbol{P} = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \implies \boldsymbol{P}^{-1} = \begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ -1 & -1 & 1 & 2 \end{pmatrix}$$

To compute the inverse of $\boldsymbol{P}$ it is enough to reduce the matrix $(\boldsymbol{P}|\boldsymbol{I})$ in row echelon form.

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \implies \left(\begin{array}{cccc|cccc} 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \end{array}\right)$$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 0 & 0 & 1 \end{array}\right) \Longrightarrow \left(\begin{array}{cccc|cccc} 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1/2 & -1/2 & -1/2 & 1/2 & 1 \end{array}\right)$$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & -1 & 1 & 2 \end{array}\right) \Longrightarrow \left(\begin{array}{cccc|cccc} 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1/2 & 1/2 & 1/2 & 1/2 & 0 \\ 0 & 0 & 1 & 1/2 & 1/2 & -1/2 & 1/2 & 0 \\ 0 & 0 & 0 & 1 & -1 & -1 & 1 & 2 \end{array}\right)$$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 1/2 & 1/2 & 1/2 & -1/2 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 & -1 & 1 & 2 \end{array}\right) \Longrightarrow \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 & -1 & 1 & 2 \end{array}\right).$$

In conclusion the matrix exponential is now

$$\begin{aligned} \boldsymbol{P}e^{\boldsymbol{D}}\boldsymbol{P}^{-1} &= \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e^{-2} & 0 & 0 & 0 \\ 0 & e^{-2} & 0 & 0 \\ 0 & 0 & e^{-2} & 0 \\ 0 & 0 & 0 & e^{3} \end{pmatrix} \begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ -1 & -1 & 1 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e^{-2} & e^{-2} & -e^{-2} & -e^{-2} \\ e^{-2} & e^{-2} & 0 & -e^{-2} \\ e^{-2} & 0 & 0 & -e^{-2} \\ -e^{3} & -e^{3} & e^{3} & 2e^{3} \end{pmatrix} \\ &= \begin{pmatrix} 2e^{-2}-e^{3} & e^{-2}-e^{3} & -e^{-2}+e^{3} & -2e^{-2}+2e^{3} \\ 0 & e^{-2} & 0 & 0 \\ 0 & 0 & e^{-2} & 0 \\ e^{-2}-e^{3} & e^{-2}-e^{3} & -e^{-2}+e^{3} & -e^{-2}+2e^{3} \end{pmatrix}. \end{aligned}$$

2.2 Solution with Cayley-Hamilton

From the knowledge of the eigenvalues one imposes a linear system for the polynomial $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3$. In this case the system is

$$\begin{aligned} p(-2) &= a_0 - 2a_1 + 4a_2 - 8a_3 &&= e^{-2} \\ p'(-2) &= a_1 + 4a_2 + 12a_3 &&= e^{-2} \\ p''(-2) &= 2a_2 - 12a_3 &&= e^{-2} \\ p(3) &= a_0 + 3a_1 + 9a_2 + 27a_3 &&= e^{3} \end{aligned}$$

which has solution

$$a_0 = \frac{477e^{-2} + 8e^3}{125} \qquad a_1 = \frac{153e^{-2} + 12e^3}{125}$$
$$a_2 = \frac{-97e^{-2} + 12e^3}{250} \qquad a_3 = \frac{-37e^{-2} + 2e^3}{250}.$$

In conclusion the required expression $e^{\boldsymbol{A}} = p(\boldsymbol{A})$, that is

$$p(\boldsymbol{A}) = a_0\boldsymbol{I} + a_1\boldsymbol{A} + a_2\boldsymbol{A}^2 + a_3\boldsymbol{A}^3$$

One can compute $\boldsymbol{A}^2$ and $\boldsymbol{A}^3$ obtains

$$\boldsymbol{A}^2 = \begin{pmatrix} -1 & -5 & 5 & 10 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ -5 & -5 & 5 & 14 \end{pmatrix} \qquad \boldsymbol{A}^3 = \begin{pmatrix} -43 & -35 & 35 & 70 \\ 0 & -8 & 0 & 0 \\ 0 & 0 & -8 & 0 \\ -35 & -35 & 35 & 62 \end{pmatrix}$$

3 Exercise 3

Solve the following optimal control problem

$$\max \int_0^1 (x - u^2)dt \qquad x' = u \qquad x(0) = 2.$$

3.1 Solution with the first variation

The Lagrangian is

$$\mathcal{L}(x, u, \lambda, \mu) = \int_0^1 [(x - u^2) - \lambda(x' - u)]dt - \mu(x(0) - 2).$$

Now perform the first variation

$$\delta\mathcal{L}(x, u, \lambda, \mu) = \frac{d}{d\alpha}\bigg|_{\alpha=0} \mathcal{L}(x + \alpha\delta_x, u + \alpha\delta_u, \lambda + \alpha\delta_\lambda, \mu + \alpha\delta_\mu)$$

Making the substitution yields

$$\delta\mathcal{L} = \lim_{\alpha\to 0}\frac{d}{d\alpha}\int_0^1 [x+\alpha\delta_x-(u+\alpha\delta_u)^2-(\lambda+\alpha\delta_\lambda)(x'+\alpha\delta_{x'}-u-\alpha\delta_u)]dt$$
$$-(\mu+\alpha\delta_\mu)(x(0)+\alpha\delta_{x(0)}-2)$$
$$= \lim_{\alpha\to 0}\int_0^1 [\delta_x-2(u+\alpha\delta_u)\delta_u-\delta_\lambda(x'+\alpha\delta_{x'}-u-\alpha\delta_u)$$
$$-(\lambda+\alpha\delta_\lambda)(\delta_{x'}-\delta_u)]dt-\delta\mu(x(0)+\alpha\delta_{x(0)}-2)-(\mu+\alpha\delta_\mu)\delta_{x(0)}$$
$$= \int_0^1 [\delta_x-2u\delta_u-\delta_\lambda(x'-u)-\lambda(\delta_{x'}-\delta_u)]dt$$
$$-\delta_\mu(x(0)-2)-\mu\delta_{x(0)}$$

One can derive directly this last passage skipping the above calculations if performs the differentiation of each variable. Now there is the problem to express the variation $\delta_{x'}$ in terms of the other variations. To see that consider the term involving $\delta_{x'}$, that is $\lambda\delta_{x'}$. If one differentiates $\lambda\delta_x$ with respect to t has

$$\frac{d}{dt}[\lambda\delta_x] = \lambda'\delta_x+\lambda\delta_{x'} \implies \lambda\delta_{x'} = \frac{d}{dt}[\lambda\delta_x]-\lambda'\delta_x.$$

Using this relation in the above expression gives

$$\delta\mathcal{L} = \int_0^1 [\delta_x-2u\delta_u-\delta_\lambda(x'-u)-\frac{d}{dt}(\lambda\delta_x)+\lambda'\delta_x+\lambda\delta_u]dt$$
$$-\delta_\mu(x(0)-2)-\mu\delta_{x(0)}.$$

Now collecting the variations in order to use the du Bois-Reymond theorem leads to

$$\delta\mathcal{L} = \int_0^1 [\delta_x(1+\lambda')+\delta_u(-2u+\lambda)+\delta_\lambda(u-x')]dt$$
$$-\lambda(1)\delta_{x(1)}+(\lambda(0)-\mu)\delta_{x(0)}-\delta\mu(x(0)-2)$$

This leads to this system of ordinary differential equations,

$$\begin{aligned} 1+\lambda' &= 0 & -\lambda(1) &= 0\\ -2u+\lambda &= 0 & \lambda(0)-\mu &= 0\\ u-x' &= 0 & x(0)-2 &= 0 \end{aligned}$$

From the first differential equation one has $\lambda = -t+c$, then using the initial condition $\lambda(1)=0$ yields $\lambda(1)=-1+c=0 \implies c=1$. So the multiplier is $\lambda(t)=-t+1$.
From the multiplier one can resolve the optimal control $u(t)$, in facts from the second differential equation $-2u+\lambda=0 \implies u(t)=\frac{\lambda(t)}{2}=\frac{-t+1}{2}$.
Finally one can reconstruct the state variable $x(t)$. From the first differential equation $x' =$

$\frac{-t+1}{2} \implies x(t) = -\frac{1}{4}t^2 + \frac{1}{2}t + c$. Now from the initial condition on $x(t)$ one has $x(0) = c = 2$. In general, to prove the fact that this control maximizes the integral is a difficult task, therefore it is better to rely on special theorems.