Connection between Laplace transform and Bode plot

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Derivation

Consider the following ODE

$$x''(t) + 3\,x'(t) + 2\,x(t) = \cos(t), \tag{1}$$

which has the general solution:

$$x(t) = \underbrace{(x(0) + x'(0))\left(2e^{-t} - e^{-2t}\right) + \frac{2}{5}e^{-2t} - \frac{1}{2}e^{-t}}_{\text{goes } \to 0 \text{ as } t \to \infty} + \frac{1}{10}\cos(t) + \frac{3}{10}\sin(t)$$

so that we can write

$$x(t) \approx \frac{1}{10}\cos(t) + \frac{3}{10}\sin(t), \qquad \text{for } t \text{ large}$$

from the identity $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$ we can deduce

$$x(t) \approx \frac{1}{10}\cos(t) + \frac{3}{10}\sin(t) = \frac{1}{\sqrt{10}}\cos(t - \phi) \qquad \text{for } t \text{ large}$$

where $\phi = \arctan(3)$. Thus we considering $\sin(t)$ as the input of the ODE (1) and $x(t)$ as the output we can say that *asymptotically* the input $\sin(t)$ is has reduced the amplitude by a factor $1/\sqrt{10}$ and shifted backward by the angle ϕ. Considering now a *generic* frequency $\cos(\omega t)$ as input we have the ODE

$$x''(t) + 3\,x'(t) + 2\,x(t) = \cos(\omega t), \tag{2}$$

which has the general solution

$$x(t) = \underbrace{A(\omega)e^{-t} + B(\omega)e^{-2t}}_{\text{goes} \to 0 \text{ as } t \to \infty} + \frac{(2-\omega^2)\cos(\omega t) + 3\omega\sin(\omega t)}{(4+\omega^2)(1+\omega^2)}$$

where

$$A(\omega) = \frac{(2\omega^4 + 10\omega^2 + 8)x_0 + (\omega^4 + 5\omega^2 + 4)x'_0 - (4+\omega^2)}{(4+\omega^2)(1+\omega^2)},$$

$$B(\omega) = \frac{2(\omega^2+1) - (\omega^4 + 5\omega^2 + 4)(x_0 + x'_0)}{(4+\omega^2)(1+\omega^2)}$$

as for the ODE (1) we can write

$$x(t) \approx \frac{(2-\omega^2)\cos(\omega t) + 3\omega\sin(\omega t)}{(4+\omega^2)(1+\omega^2)}, \qquad \text{for } t \text{ large}$$

from the identity $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$ we can deduce

$$x(t) \approx \frac{(2-\omega^2)\cos(\omega t) + 3\omega\sin(\omega t)}{(4+\omega^2)(1+\omega^2)} = \frac{\cos(\omega t - \phi)}{\sqrt{(4+\omega^2)(1+\omega^2)}} \qquad \text{for } t \text{ large}$$

where $\phi = \arctan(3\omega/(2-\omega^2))$. Thus we considering $\sin(t)$ as the input of the ODE (2) and $x(t)$ as the output we can say that *asymptotically* the input $\sin(t)$ is has reduced the amplitude by a factor $1/\sqrt{(4+\omega^2)(1+\omega^2)}$ and shifted backward by the angle ϕ.

Generalization Consider now the ODE

$$x^{(N)} + \sum_{k=0}^{N-1} a_k x^{(k)}(t) = \cos(\omega t), \quad x(0) = x'(0) = \cdots = x^{N-1}(0) = 0, \quad (3)$$

the response $x(t)$ to the input $\cos(\omega t)$ is in general

$$x(t) = x_0(t) + A\cos(\omega t + \phi),$$

where $x_0(t) \to 0$ if $t \to \infty$ if the homogenous ODE $\sum_{k=0}^{N} a_k x^{(k)}(t) = 0$ is stable. In this case we say that the signal $\cos(\omega t)$ is gained o reduced by a factor A and shifted by an angle ϕ. If we take the Laplace transform of (3) we have

$$\left(s^N + \sum_{k=0}^{N-1} a_k s^k\right) x(s) = \frac{s}{s^2+\omega^2},$$

and if $s_0,\ s_1,\ \ldots,\ s_{N-1}$ are the root of the polynomial $s^N + \sum_{k=0}^{N-1} a_k s^k$ we have

$$\prod_{k=0}^{N-1} (s - s^k) x(s) = \frac{s}{s^2+\omega^2},$$

and thus,

$$x(s) = G(s) \times \frac{s}{s^2+\omega^2} \tag{4}$$

where

$$G(s) = \frac{1}{\prod_{k=0}^{N-1}(s - s^k)}$$

is the transfer function of the ODE. Using simple fraction expansion and for simplicity assuming all the root simple we have

$$x(s) = \frac{A(\omega)s - B(\omega)\omega}{s^2+\omega^2} + \sum_{k=0}^{N} \frac{C_k}{s - s^k} \tag{5}$$

where $A(\omega)$, $B(\omega)$ and C_k are computed by equating (4) with (5). Reversing Laplace transform form $x(s)$ we have (remember $\Re(s_k) < 0$ because ODE is stable):

$$x(t) = K(\omega)\cos(\omega t + \phi(\omega)) + \underbrace{\sum_{k=0}^{N} C_k e^{ts^k}}_{\text{goes} \to 0 \text{ as } t \to \infty}$$

where

$$K(\omega) = \sqrt{A(\omega)^2 + B(\omega)^2}, \qquad \phi(\omega) = \arctan\left(\frac{B(\omega)}{A(\omega)}\right). \tag{6}$$

To compute $K(\omega)$ and $\phi(\omega)$ we need to compute $A(\omega)$ and $B(\omega)$. To this purpose from (4) and (5) we have

$$G(s) \times \frac{s}{s^2+\omega^2} = \frac{A(\omega)s - B(\omega)\omega}{s^2+\omega^2} + \sum_{k=0}^{N} \frac{C_k}{s-s^k}$$

multiply both side by $s - \imath\,\omega$ we have

$$G(s) \times \frac{s}{s+\imath\,\omega} = \frac{A(\omega)s - B(\omega)\omega}{s+\imath\,\omega} + (s-\imath\,\omega)\sum_{k=0}^{N} \frac{C_k}{s-s^k}$$

and computing equation in $s = \imath\,\omega$ we have

$$G(\imath\,\omega) \times \frac{1}{2} = \frac{A(\omega)\imath\,\omega - B(\omega)\omega}{2\imath\,\omega} + 0 \times \sum_{k=0}^{N} \frac{C_k}{s-s^k}$$

and thus

$$G(\imath\,\omega) = A(\omega) - \frac{B(\omega)}{\imath} = A(\omega) + \imath\,B(\omega)$$

and from (6) we have

$$\begin{aligned} K(\omega) &= \sqrt{A(\omega)^2 + B(\omega)^2} = |G(\imath\,\omega)|\,, \\ \phi(\omega) &= \arctan\left(\frac{B(\omega)}{A(\omega)}\right) = \arctan\left(\frac{\Im(G(\imath\,\omega))}{\Re(G(\imath\,\omega))}\right). \end{aligned}$$