Numerical Methods for Dynamic System and Control del 14/1/2013

Surname | | Name | | Mat. number | |

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1

Study the following constrained minimization problem.

$$\text{minimize} \quad f(x,y,z) = x + y^2 - z, \qquad \text{subject to} \qquad xy + z = 1, \qquad 0 \le x \le 1,$$

KKT system of first order condition:

$$\begin{cases} 1 - \lambda y - \mu_1 + \mu_2 &= 0 \\ 2y - \lambda x &= 0 \\ -1 - \lambda &= 0 \\ xy + z - 1 &= 0 \\ \mu_1 x &= 0 \\ \mu_2(1-x) &= 0 \end{cases}$$

Solutions of KKT system:

$$\begin{cases} x = 0, \quad y = 0, \quad z = 1, \quad \lambda = -1, \quad \mu_1 = 1, \quad \mu_2 = 0 & \text{OK} \\ x = 2, \quad y = -1, \quad z = 3, \quad \lambda = -1, \quad \mu_1 = 0, \quad \mu_2 = 0 & \text{NO } (x > 1) \\ x = 1, \quad y = -1/2, \quad z = 3/2, \quad \lambda = -1, \quad \mu_1 = 0, \quad \mu_2 = -1/2 & \text{NO } (\mu_2 < 0) \end{cases}$$

Discussion of the stationary point: $x = 0, \quad y = 0, \quad z = 1, \quad \lambda = -1, \quad \mu_1 = 1, \quad \mu_2 = 0$

$$\nabla[h, g_1] = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \qquad \boldsymbol{K} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad \boldsymbol{K}^T \nabla^2 \mathcal{L} \boldsymbol{K} = [2]$$

2

Solve the following recurrence

$$x_{k+1} = x_k + k, \qquad x_0 = 0$$
$$y_{k+1} = x_k + y_k, \qquad y_0 = 1,$$

$\mathcal{Z}-$transform:

$$\begin{aligned}(\zeta-1)x(\zeta) &= \frac{\zeta}{(\zeta-1)^2},\\ (\zeta-1)y(\zeta) &= \zeta + x(\zeta),\end{aligned}$$

Solution in $\mathcal{Z}$:

$$\begin{aligned}x(\zeta) &= \frac{\zeta}{(\zeta-1)^2},\\ y(\zeta) &= \frac{\zeta}{(\zeta-1)^4} + \frac{\zeta}{\zeta-1},\end{aligned}$$

Solution in k

$$\begin{aligned}x_k &= \tfrac{1}{2}k(k-1),\\ y_k &= \tfrac{1}{6}(k+1)(k^2-4k+6) = 1 + \frac{k}{3} - \frac{k^2}{2} + \frac{k^3}{6},\end{aligned}$$

3

Solve using Laplace transform the following boundary value problem

$$x''(t) + y(t) = e^t + e^{-t}$$
$$y''(t) - y(t) = 0$$
$$x(0) = 1, \quad y(0) = 1, \quad x(1) = \frac{1}{e}, \quad y(1) = e,$$

Suggestion: Set $A = x'(0)$ and $B = y'(0)$ then using Laplace transform compute A and B which satisfy the boundary conditions.

The Laplace transform:

$$\begin{cases} s^2 x(s) - A - s + y(s) &= \dfrac{2s}{s^2-1} \\ s^2 y(s) - B - s - y(s) &= 0 \end{cases}$$

Solution in s with partial fraction expansion:

$$\begin{cases} x(s) = \dfrac{s^3 - B + As^2 - A}{s^2(s^2-1)} = \dfrac{A+B}{s^2} + \dfrac{1+B}{2(s+1)} + \dfrac{1-B}{2(s-1)} \\ y(s) = \dfrac{B+s}{s^2-1} = \dfrac{1-B}{2(s+1)} + \dfrac{1+B}{2(s-1)} \end{cases}$$

Value of A and B:

$A = -1, \qquad B = +1$

Solution in t:

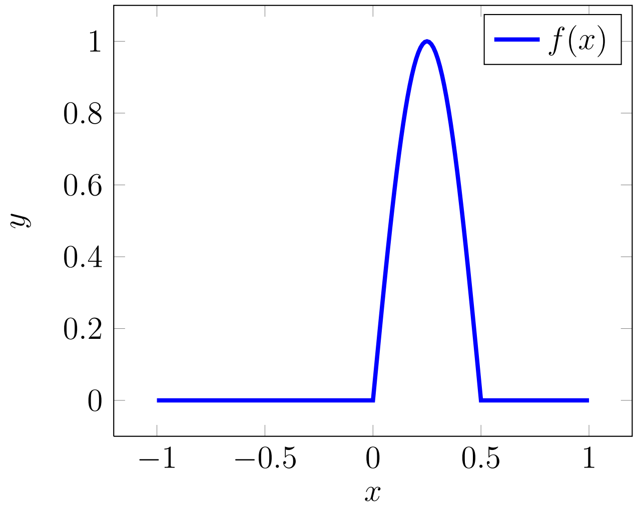
$$\begin{cases} x(t) = e^{-t} \\ y(t) = e^t \end{cases}$$

4

Compute the coefficients of the Fourier series of the following function:

$$f(x) = \begin{cases} 0 & \text{for } -1 \le x < 0 \\ \sin(2\pi x) & \text{for } 0 \le x < 1/2 \\ 0 & \text{for } 1/2 \le x < 1 \end{cases}$$

defined for $x \in (-1, 1)$ and extended periodically.



Coefficients a_k and b_k for the Fourier serie:

$$a_0 = \frac{1}{\pi}$$

$$a_2 = 0, \qquad \text{for } k \neq 2 \qquad a_k = -\frac{2}{\pi(k^2-4)}\left(1 + \cos\left(\frac{\pi k}{2}\right)\right) = \frac{-2}{\pi(k^2-4)} \begin{cases} 2 & \text{for } k = 4m \\ 1 & \text{for } k = 4m+1 \\ 0 & \text{for } k = 4m+2 \\ 1 & \text{for } k = 4m+3 \end{cases}$$

$$b_2 = \frac{1}{4}, \qquad \text{for } k \neq 2 \qquad b_k = -\frac{2}{\pi(k^2-4)} \sin\left(\frac{\pi k}{2}\right) = \frac{2}{\pi(k^2-4)} \begin{cases} 0 & \text{for } k = 4m \\ -1 & \text{for } k = 4m+1 \\ 0 & \text{for } k = 4m+2 \\ 1 & \text{for } k = 4m+3 \end{cases}$$