Numerical Methods for Dynamic System and Control del 7/2/2013

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1

Study the following constrained minimization problem.

$$\text{minimize} \quad f(x,y,z) = y\,z^2 + x, \qquad \text{subject to} \qquad z = y, \qquad y \geq 0, \qquad x + y \geq 1,$$

KKT system of first order condition:

$$\begin{cases} 1-\mu_2 &=& 0 \\ z^2+\lambda-\mu_1-\mu_2 &=& 0 \\ 2yz-\lambda &=& 0 \\ z-y &=& 0 \\ \mu_1 y &=& 0 \\ \mu_2(x+y-1) &=& 0 \end{cases}$$

Solutions of KKT system:

$$\begin{cases} x=1, \quad y=0, \quad z=0, \quad \lambda=0, \quad \mu_1=-1, \quad \mu_2=1 & \text{NO } (\mu_1<0) \\ x=1-\dfrac{1}{\sqrt{3}}, \quad y=\dfrac{1}{\sqrt{3}}, \quad z=\dfrac{1}{\sqrt{3}}, \quad \lambda=\dfrac{2}{3}, \quad \mu_1=0, \quad \mu_2=1 & \text{OK} \\ x=1+\dfrac{1}{\sqrt{3}}, \quad y=-\dfrac{1}{\sqrt{3}}, \quad z=-\dfrac{1}{\sqrt{3}}, \quad \lambda=\dfrac{2}{3}, \quad \mu_1=0, \quad \mu_2=1 & \text{NO (contraint violation)} \end{cases}$$

Discussion of the stationary point: $x = 1-\dfrac{1}{\sqrt{3}}, \quad y=\dfrac{1}{\sqrt{3}}, \quad z=\dfrac{1}{\sqrt{3}}, \quad \lambda=\dfrac{2}{3}, \quad \mu_1=0, \quad \mu_2=1$

$$\nabla[h,g_2] = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 1 & 0 \end{pmatrix} \qquad \boldsymbol{K} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \qquad \boldsymbol{K}^T \nabla^2 \mathcal{L} \boldsymbol{K} = [2\sqrt{3}]$$

2

Solve the following recurrence

$$x_{k+1} = x_k + k^2, \qquad x_0 = 0$$
$$y_{k+1} = x_k - y_k, \qquad y_0 = 1,$$

$\mathcal{Z}-$transform:

$$(\zeta - 1)x(\zeta) = \frac{\zeta(\zeta+1)}{(\zeta-1)^3},$$
$$(\zeta + 1)y(\zeta) = \zeta + x(\zeta),$$

Solution in $\mathcal{Z}$:

$$x(\zeta) = \frac{\zeta(\zeta+1)}{(\zeta-1)^4},$$
$$y(\zeta) = \frac{\zeta}{(\zeta-1)^4} + \frac{\zeta}{\zeta+1},$$

Solution in k

$$x_k = \frac{1}{3}k^3 - \frac{1}{2}k^2 + \frac{1}{6}k,$$
$$y_k = \tfrac{1}{6}k^3 - \frac{1}{2}k^2 + \frac{1}{3}k + (-1)^k,$$

3

Solve using Laplace transform the following boundary value problem

$$x''(t) + y'(t) = 3$$
$$y''(t) - y(t) = -t$$
$$x(0) = 0, \quad y(0) = 0, \quad x(1) = 1, \quad y(1) = 1,$$

Suggestion: Set $A = x'(0)$ and $B = y'(0)$ then using Laplace transform compute A and B which satisfy the boundary conditions.

The Laplace transform:

$$\begin{cases} s^2 x(s) - A + s y(s) &= \dfrac{3}{s} \\ s^2 y(s) - B - y(s) &= -\dfrac{1}{s^2} \end{cases}$$

Solution in s with partial fraction expansion:

$$\begin{cases} x(s) &= \dfrac{A}{s^2} - \dfrac{B}{s(s^2-1)} + \dfrac{3s^2-2}{s^3(s^2-1)} &= \dfrac{2}{s^3} + \dfrac{A}{s^2} + \dfrac{B-1}{s} + \dfrac{1-B}{2(s+1)} + \dfrac{1-B}{2(s-1)} \\ y(s) &= \dfrac{B}{s^2-1} - \dfrac{1}{s^2(s^2-1)} &= \dfrac{1}{s^2} + \dfrac{1-B}{2(s+1)} - \dfrac{1-B}{2(s-1)} \end{cases}$$

Value of A and B:

$A = 0, \qquad B = 1$

Solution in t:

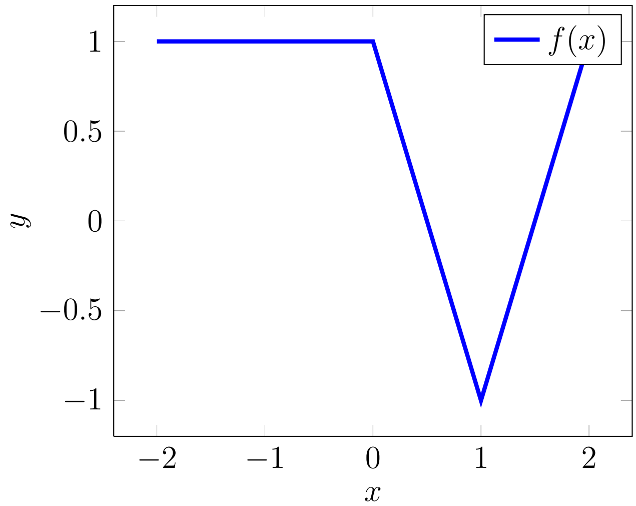
$$\begin{cases} x(t) &= t^2 \\ y(t) &= t \end{cases}$$

4

Compute the coefficients of the Fourier series of the following function:

$$f(x) = \begin{cases} 1 & \text{for } -2 \leq x < 0 \\ 1 - 2x & \text{for } 0 \leq x < 1 \\ 2x - 3 & \text{for } 1 \leq x < 2 \end{cases}$$

defined for $x \in (-2, 2)$ and extended periodically.



Coefficients a_k and b_k for the Fourier serie:

$a_0 = 1$

$$a_k = \frac{4}{\pi^2 k^2}\left(1 - 2\cos\left(\frac{\pi k}{2}\right) + (-1)^k\right) = \frac{16}{\pi^2 k^2}\begin{cases} 1 & \text{for } k = 4m + 2 \\ 0 & \text{otherwise} \end{cases}$$

$$b_k = -\frac{8}{\pi^2 k^2}\sin\left(\frac{\pi k}{2}\right) = \frac{8}{\pi^2 k^2}\begin{cases} 0 & \text{for } k = 4m \\ -1 & \text{for } k = 4m + 1 \\ 0 & \text{for } k = 4m + 2 \\ 1 & \text{for } k = 4m + 3 \end{cases}$$