Examples of Laplace inversion using partial fraction (with and without complex exp)

Error, missing operator or `;`

```
> restart:
> with(inttrans):
```

simple root case

The rational polynomial

```
> P := 1+s ;
  Q := 1+s^2 ;
  R := P/Q ;
```

$$P := 1 + s$$
$$Q := 1 + s^2$$
$$R := \frac{1+s}{1+s^2} \quad (1.1)$$

Factor Q in linear factors

```
> roots_of_Q_sols := solve( Q, {s} ) ;
```

$$roots\_of\_Q\_sols := \{s = I\}, \{s = -I\} \quad (1.2)$$

```
> n_roots := nops({roots_of_Q_sols}) ;
```

$$n\_roots := 2 \quad (1.3)$$

extract the roots

```
> roots_of_Q := <seq( subs(roots_of_Q_sols[i],s),i=1..n_roots) >;
```

$$roots\_of\_Q := \begin{bmatrix} I \\ -I \end{bmatrix} \quad (1.4)$$

The Q factorized

```
> Q_factored := mul( s-roots_of_Q[i], i=1..n_roots);
```

$$Q\_factored := (s - I)\,(s + I) \quad (1.5)$$

The generic partial fraction expansion

```
> faction_expansion := add( alpha||i/(s-roots_of_Q[i]), i=1..
  n_roots);
```

$$faction\_expansion := \frac{\alpha 1}{s - I} + \frac{\alpha 2}{s + I} \quad (1.6)$$

Find the coefficients alpha's

```
> sols_alphas := seq( alpha||i = subs( s=roots_of_Q[i],
  P/Q_factored*(s-roots_of_Q[i]) ), i=1..n_roots ) ;
```

$$sols\_alphas := \alpha 1 = \frac{1}{2} - \frac{1}{2} I, \alpha 2 = \frac{1}{2} + \frac{1}{2} I \quad (1.7)$$

```
> alphas := <seq( subs(sols_alphas[i],alpha||i),i=1..n_roots) >;
```

$$alphas := \begin{bmatrix} \frac{1}{2} - \frac{1}{2} I \\ \frac{1}{2} + \frac{1}{2} I \end{bmatrix} \quad (1.8)$$

The expansion of R=P/Q

```
> P_over_Q_expanded := subs(sols_alphas,faction_expansion) ;
```

$$P\_over\_Q\_expanded := \frac{\frac{1}{2} - \frac{1}{2} I}{s - I} + \frac{\frac{1}{2} + \frac{1}{2} I}{s + I} \quad (1.9)$$

Formally the inversion of Laplace tranform (of complex function = not well defined!!!)

```
> P_over_Q_inverted := add( alphas[i] * exp(roots_of_Q[i]*t), i=1..
  n_roots) ;
```

$$P\_over\_Q\_inverted := \left(\frac{1}{2} - \frac{1}{2} I\right) e^{I t} + \left(\frac{1}{2} + \frac{1}{2} I\right) e^{-I t} \quad (1.10)$$

Expand complex number and simplify

```
> evalc(P_over_Q_inverted);
```

$$\cos(t) + \sin(t) \quad (1.11)$$

```
> invlaplace( R, s, t ) ;
```

$$\cos(t) + \sin(t) \quad (1.12)$$

multiple root case

The rational polynomial

```
> P := 1+s ;
  Q := (1+s^2)^2 ;
  R := P/Q ;
```

$$P := 1 + s$$

$$Q := (1 + s^2)^2$$

$$R := \frac{1 + s}{(1 + s^2)^2} \quad (2.1)$$

Factor Q in linear factors

```
> roots_of_Q_sols := solve( Q, {s} ) ;
```

$$roots\_of\_Q\_sols := \{s = I\}, \{s = -I\}, \{s = I\}, \{s = -I\} \quad (2.2)$$

```
> n_roots := nops({roots_of_Q_sols}) ;
```

$$n\_roots := 2 \quad (2.3)$$

```
> mult := 2 ; # the multiuplicity of the roots
```

$$mult := 2 \quad (2.4)$$

extract the roots

```
> roots_of_Q := <seq( subs(roots_of_Q_sols[i],s),i=1..n_roots) >;
```

$$roots\_of\_Q := \begin{bmatrix} I \\ -I \end{bmatrix} \quad (2.5)$$

The Q factorized

```
> Q_factored := mul( s-roots_of_Q[i], i=1..n_roots)^mult ;
```

$$Q\_factored := (s - I)^2 \, (s + I)^2 \quad (2.6)$$

The generic partial fraction expansion

```
> faction_expansion_power_1 := add( alpha||i/(s-roots_of_Q[i]), i=
  1..n_roots) ;
  faction_expansion_power_2 := add( alpha||(i+2)/(s-roots_of_Q[i])
  ^2, i=1..n_roots) ;

  faction_expansion := faction_expansion_power_1 +
  faction_expansion_power_2;
```

$$faction\_expansion\_power\_1 := \frac{\alpha 1}{s - I} + \frac{\alpha 2}{s + I}$$

$$faction\_expansion\_power\_2 := \frac{\alpha 3}{(s - I)^2} + \frac{\alpha 4}{(s + I)^2}$$

$$faction\_expansion := \frac{\alpha 1}{s - I} + \frac{\alpha 2}{s + I} + \frac{\alpha 3}{(s - I)^2} + \frac{\alpha 4}{(s + I)^2} \quad (2.7)$$

Find the coefficients alpha's

```
> sols_alphas := seq( alpha||i = subs( s=roots_of_Q[i],diff
  (P/Q_factored*(s-roots_of_Q[i])^2,s) ), i=1..n_roots ),
  seq( alpha||(i+2) = subs( s=roots_of_Q[i],P/Q_factored*(s-
  roots_of_Q[i])^2 ), i=1..n_roots ) ;
```

$$sols\_alphas := \alpha 1 = -\frac{1}{4} I, \alpha 2 = \frac{1}{4} I, \alpha 3 = -\frac{1}{4} - \frac{1}{4} I, \alpha 4 = -\frac{1}{4} + \frac{1}{4} I \quad (2.8)$$

```
> alphas := <seq( subs(sols_alphas[i],alpha||i),i=1..mult*n_roots)
  >;
```

$$alphas := \begin{bmatrix} -\frac{1}{4} I \\ \frac{1}{4} I \\ -\frac{1}{4} - \frac{1}{4} I \\ -\frac{1}{4} + \frac{1}{4} I \end{bmatrix} \quad (2.9)$$

The expansion of R=P/Q

```
> P_over_Q_expanded_power_1 := subs(sols_alphas,
  faction_expansion_power_1) ;
  P_over_Q_expanded_power_2 := subs(sols_alphas,
  faction_expansion_power_2) ;
  P_over_Q_expanded         := P_over_Q_expanded_power_1 +
  P_over_Q_expanded_power_2 ;
```

$$P\_over\_Q\_expanded\_power\_1 := -\frac{\frac{1}{4}\,\mathrm{I}}{s-\mathrm{I}} + \frac{\frac{1}{4}\,\mathrm{I}}{s+\mathrm{I}}$$

$$P\_over\_Q\_expanded\_power\_2 := \frac{-\frac{1}{4}-\frac{1}{4}\,\mathrm{I}}{(s-\mathrm{I})^2} + \frac{-\frac{1}{4}+\frac{1}{4}\,\mathrm{I}}{(s+\mathrm{I})^2}$$

$$P\_over\_Q\_expanded := -\frac{\frac{1}{4}\,\mathrm{I}}{s-\mathrm{I}} + \frac{\frac{1}{4}\,\mathrm{I}}{s+\mathrm{I}} + \frac{-\frac{1}{4}-\frac{1}{4}\,\mathrm{I}}{(s-\mathrm{I})^2} + \frac{-\frac{1}{4}+\frac{1}{4}\,\mathrm{I}}{(s+\mathrm{I})^2} \quad \mathbf{(2.10)}$$

Check for correctness

```
> simplify(R-P_over_Q_expanded) ;
```

$$0 \quad \mathbf{(2.11)}$$

Inversion of fraction with power = 1

```
> P_over_Q_inverted_power_1 := add( alphas[i] * exp(roots_of_Q[i]*
  t), i=1..n_roots) ;
```

$$P\_over\_Q\_inverted\_power\_1 := -\frac{1}{4}\,\mathrm{I}\,\mathrm{e}^{\mathrm{I}\,t} + \frac{1}{4}\,\mathrm{I}\,\mathrm{e}^{-\mathrm{I}\,t} \quad \mathbf{(2.12)}$$

```
> P_over_Q_expanded_power_2_integrated := int
  (P_over_Q_expanded_power_2,s) ;
```

$$P\_over\_Q\_expanded\_power\_2\_integrated := \frac{\frac{1}{4}+\frac{1}{4}\,\mathrm{I}}{s-\mathrm{I}} + \frac{\frac{1}{4}-\frac{1}{4}\,\mathrm{I}}{s+\mathrm{I}} \quad \mathbf{(2.13)}$$

```
> P_over_Q_inverted_power_2 := add( t * alphas[i+2] * exp
  (roots_of_Q[i]*t), i=1..n_roots) ;
```

$$P\_over\_Q\_inverted\_power\_2 := \left(-\frac{1}{4}-\frac{1}{4}\,\mathrm{I}\right) t\,\mathrm{e}^{\mathrm{I}\,t} + \left(-\frac{1}{4}+\frac{1}{4}\,\mathrm{I}\right) t\,\mathrm{e}^{-\mathrm{I}\,t} \quad \mathbf{(2.14)}$$

Expand complex number and simplify

```
> evalc(P_over_Q_inverted_power_1+P_over_Q_inverted_power_2);
```

$$\frac{1}{2}\sin(t) - \frac{1}{2}\,t\cos(t) + \frac{1}{2}\,t\sin(t) \quad \mathbf{(2.15)}$$

```
> invlaplace( R, s, t) ;
```

$$-\frac{1}{2}\,t\cos(t) + \frac{1}{2}\sin(t)\,(1+t) \quad \mathbf{(2.16)}$$

multiple root case

The rational polynomial

```
> P := 1+s+s ;
  Q := (1+2*s+2*s^2)^3 ;
  R := P/Q ;
```

$$P := 1 + 2\,s$$

$$Q := \left(1 + 2\,s + 2\,s^2\right)^3$$

$$R := \frac{1 + 2\,s}{\left(1 + 2\,s + 2\,s^2\right)^3} \tag{3.1}$$

Factor Q in linear factors

```
> roots_of_Q_sols := solve( Q, {s} ) ;
```

$$roots\_of\_Q\_sols := \left\{s = -\frac{1}{2} + \frac{1}{2}\,I\right\}, \left\{s = -\frac{1}{2} - \frac{1}{2}\,I\right\}, \left\{s = -\frac{1}{2} + \frac{1}{2}\,I\right\}, \left\{s = -\frac{1}{2} - \frac{1}{2}\,I\right\}, \left\{s = -\frac{1}{2} + \frac{1}{2}\,I\right\}, \left\{s = -\frac{1}{2} - \frac{1}{2}\,I\right\} \tag{3.2}$$

```
> n_roots := nops({roots_of_Q_sols}) ;
```

$$n\_roots := 2 \tag{3.3}$$

```
> mult := 3 ; # the multiuplicity of the roots
```

$$mult := 3 \tag{3.4}$$

extract the roots

```
> roots_of_Q := <seq( subs(roots_of_Q_sols[i],s),i=1..n_roots) >;
```

$$roots\_of\_Q := \begin{bmatrix} -\frac{1}{2} + \frac{1}{2}\,I \\ -\frac{1}{2} - \frac{1}{2}\,I \end{bmatrix} \tag{3.5}$$

The Q factorized

```
> Q_factored := mul( s-roots_of_Q[i], i=1..n_roots)^mult ;
```

$$Q\_factored := \left(s + \frac{1}{2} - \frac{1}{2}\,I\right)^3 \left(s + \frac{1}{2} + \frac{1}{2}\,I\right)^3 \tag{3.6}$$

The generic partial fraction expansion

```
> faction_expansion_power_1 := add( alpha||i/(s-roots_of_Q[i]), i=
  1..n_roots) ;
  faction_expansion_power_2 := add( alpha||(i+2)/(s-roots_of_Q[i])
  ^2, i=1..n_roots) ;
  faction_expansion_power_3 := add( alpha||(i+4)/(s-roots_of_Q[i])
  ^3, i=1..n_roots) ;
  faction_expansion := faction_expansion_power_1 +
  faction_expansion_power_2+ faction_expansion_power_3;
```

$$faction\_expansion\_power\_1 := \frac{\alpha 1}{s + \frac{1}{2} - \frac{1}{2}\,I} + \frac{\alpha 2}{s + \frac{1}{2} + \frac{1}{2}\,I}$$

$$faction\_expansion\_power\_2 := \frac{\alpha 3}{\left(s + \frac{1}{2} - \frac{1}{2}\,I\right)^2} + \frac{\alpha 4}{\left(s + \frac{1}{2} + \frac{1}{2}\,I\right)^2}$$

$$faction\_expansion\_power\_3 := \frac{\alpha 5}{\left(s + \frac{1}{2} - \frac{1}{2}\,I\right)^3} + \frac{\alpha 6}{\left(s + \frac{1}{2} + \frac{1}{2}\,I\right)^3}$$

$$faction\_expansion := \frac{\alpha 1}{s+\frac{1}{2}-\frac{1}{2}\mathrm{I}} + \frac{\alpha 2}{s+\frac{1}{2}+\frac{1}{2}\mathrm{I}} + \frac{\alpha 3}{\left(s+\frac{1}{2}-\frac{1}{2}\mathrm{I}\right)^2} + \frac{\alpha 4}{\left(s+\frac{1}{2}+\frac{1}{2}\mathrm{I}\right)^2} + \frac{\alpha 5}{\left(s+\frac{1}{2}-\frac{1}{2}\mathrm{I}\right)^3} + \frac{\alpha 6}{\left(s+\frac{1}{2}+\frac{1}{2}\mathrm{I}\right)^3} \tag{3.7}$$

```
> EQ1 := subs(s=0,numer(faction_expansion-R)) ;
  EQ2 := subs(s=0,diff(numer(faction_expansion-R),s)) ;
  EQ3 := subs(s=0,diff(numer(faction_expansion-R),s,s)) ;
  EQ4 := subs(s=0,diff(numer(faction_expansion-R),s,s,s)) ;
  EQ5 := subs(s=0,diff(numer(faction_expansion-R),s,s,s,s)) ;
  EQ6 := subs(s=0,diff(numer(faction_expansion-R),s,s,s,s,s)) ;
```

$$EQ1 := 8 - 8\,\alpha 1 - 8\,\alpha 2 + 16\,\mathrm{I}\,\alpha 6 - 16\,\mathrm{I}\,\alpha 5 + 16\,\mathrm{I}\,\alpha 4 - 16\,\mathrm{I}\,\alpha 3 + 8\,\mathrm{I}\,\alpha 2 - 8\,\mathrm{I}\,\alpha 1 + 16\,\alpha 5 + 16\,\alpha 6$$

$$EQ2 := 64 - 96\,\alpha 1 - 96\,\alpha 2 + 160\,\mathrm{I}\,\alpha 4 - 192\,\mathrm{I}\,\alpha 5 - 80\,\mathrm{I}\,\alpha 1 + 80\,\mathrm{I}\,\alpha 2 + 192\,\mathrm{I}\,\alpha 6 - 160\,\mathrm{I}\,\alpha 3 + 96\,\alpha 5 + 96\,\alpha 6 - 32\,\alpha 3 - 32\,\alpha 4$$

$$EQ3 := 480 - 1120\,\alpha 1 - 1120\,\alpha 2 + 1536\,\mathrm{I}\,\alpha 4 - 800\,\mathrm{I}\,\alpha 1 + 800\,\mathrm{I}\,\alpha 2 - 1920\,\mathrm{I}\,\alpha 5 + 1920\,\mathrm{I}\,\alpha 6 - 1536\,\mathrm{I}\,\alpha 3 + 384\,\alpha 5 + 384\,\alpha 6 - 576\,\alpha 3 - 576\,\alpha 4$$

$$EQ4 := 3264 - 12480\,\alpha 1 - 12480\,\alpha 2 + 13824\,\mathrm{I}\,\alpha 4 - 7680\,\mathrm{I}\,\alpha 1 + 7680\,\mathrm{I}\,\alpha 2 - 16896\,\mathrm{I}\,\alpha 5 + 16896\,\mathrm{I}\,\alpha 6 - 13824\,\mathrm{I}\,\alpha 3 - 768\,\alpha 5 - 768\,\alpha 6 - 7680\,\alpha 3 - 7680\,\alpha 4$$

$$EQ5 := 19200 - 130560\,\alpha 1 - 130560\,\alpha 2 - 129024\,\mathrm{I}\,\alpha 5 + 129024\,\mathrm{I}\,\alpha 6 - 113664\,\mathrm{I}\,\alpha 3 + 113664\,\mathrm{I}\,\alpha 4 - 69120\,\mathrm{I}\,\alpha 1 + 69120\,\mathrm{I}\,\alpha 2 - 36864\,\alpha 5 - 36864\,\alpha 6 - 86016\,\alpha 3 - 86016\,\alpha 4$$

$$EQ6 := 92160 - 1259520\,\alpha 1 - 1259520\,\alpha 2 + 568320\,\mathrm{I}\,\alpha 2 - 829440\,\mathrm{I}\,\alpha 5 + 829440\,\mathrm{I}\,\alpha 6 - 829440\,\mathrm{I}\,\alpha 3 + 829440\,\mathrm{I}\,\alpha 4 - 568320\,\mathrm{I}\,\alpha 1 - 460800\,\alpha 5 - 460800\,\alpha 6 - 829440\,\alpha 3 - 829440\,\alpha 4 \tag{3.8}$$

```
> sols_alphas := solve( {EQ||(1..6)}, {alpha||(1..6)} );
```

$$sols\_alphas := \left\{\alpha 1 = 0, \alpha 2 = 0, \alpha 3 = -\frac{1}{8}\,\mathrm{I}, \alpha 4 = \frac{1}{8}\,\mathrm{I}, \alpha 5 = -\frac{1}{8}, \alpha 6 = -\frac{1}{8}\right\} \tag{3.9}$$

Find the coefficients alpha's

```
> sols_alphas := seq( alpha||i = subs( s=roots_of_Q[i],diff
  (P/Q_factored*(s-roots_of_Q[i])^3,s,s) ), i=1..n_roots ),
  seq( alpha||(i+2) = subs( s=roots_of_Q[i],diff(P/Q_factored*(s-
  roots_of_Q[i])^3,s) ), i=1..n_roots )
  ,
  seq( alpha||(i+4) = subs( s=roots_of_Q[i],P/Q_factored*(s-
  roots_of_Q[i])^3 ), i=1..n_roots ) ;
```

$$sols\_alphas := \alpha 1 = 0, \alpha 2 = 0, \alpha 3 = -\mathrm{I}, \alpha 4 = \mathrm{I}, \alpha 5 = -1, \alpha 6 = -1 \tag{3.10}$$

```
> alphas := <seq( subs(sols_alphas[i],alpha||i),i=1..mult*n_roots)
  >;
```

$$alphas := \begin{bmatrix} 0 \\ 0 \\ -I \\ I \\ -1 \\ -1 \end{bmatrix} \quad (3.11)$$

The expansion of R=P/Q

```
> P_over_Q_expanded_power_1 := subs(sols_alphas,
  faction_expansion_power_1) ;
  P_over_Q_expanded_power_2 := subs(sols_alphas,
  faction_expansion_power_2) ;
  P_over_Q_expanded_power_3 := subs(sols_alphas,
  faction_expansion_power_3) ;
  P_over_Q_expanded         := P_over_Q_expanded_power_1 +
  P_over_Q_expanded_power_2 +P_over_Q_expanded_power_3 ;
```

$$P\_over\_Q\_expanded\_power\_1 := 0$$

$$P\_over\_Q\_expanded\_power\_2 := -\frac{I}{\left(s+\frac{1}{2}-\frac{1}{2}I\right)^2}+\frac{I}{\left(s+\frac{1}{2}+\frac{1}{2}I\right)^2}$$

$$P\_over\_Q\_expanded\_power\_3 := -\frac{1}{\left(s+\frac{1}{2}-\frac{1}{2}I\right)^3}-\frac{1}{\left(s+\frac{1}{2}+\frac{1}{2}I\right)^3}$$

$$P\_over\_Q\_expanded := -\frac{I}{\left(s+\frac{1}{2}-\frac{1}{2}I\right)^2}+\frac{I}{\left(s+\frac{1}{2}+\frac{1}{2}I\right)^2} - \frac{1}{\left(s+\frac{1}{2}-\frac{1}{2}I\right)^3}-\frac{1}{\left(s+\frac{1}{2}+\frac{1}{2}I\right)^3} \quad (3.12)$$

Check for correctness

```
> simplify(R-P_over_Q_expanded) ;
```

$$\frac{56\,(1+2\,s)}{(2\,s+1+I)^3\,(-2\,s-1+I)^3} \quad (3.13)$$

Inversion of fraction with power = 1

```
> P_over_Q_inverted_power_1 := add( alphas[i] * exp(roots_of_Q[i]*
  t), i=1..n_roots) ;
```

$$P\_over\_Q\_inverted\_power\_1 := 0 \quad (3.14)$$

```
> P_over_Q_expanded_power_2_integrated := int
  (P_over_Q_expanded_power_2,s) ;
  P_over_Q_inverted_power_2 := add( t * alphas[i+2] * exp
  (roots_of_Q[i]*t), i=1..n_roots) ;
```

$$P\_over\_Q\_expanded\_power\_2\_integrated := \frac{I}{s+\frac{1}{2}-\frac{1}{2}I} - \frac{I}{s+\frac{1}{2}+\frac{1}{2}I}$$

$$P\_over\_Q\_inverted\_power\_2 := -I\,t\,e^{\left(-\frac{1}{2}+\frac{1}{2}I\right)t} + I\,t\,e^{\left(-\frac{1}{2}-\frac{1}{2}I\right)t} \quad \textbf{(3.15)}$$

```
> P_over_Q_expanded_power_3_integrated := int(int
  (P_over_Q_expanded_power_3,s),s) ;
  P_over_Q_inverted_power_3 := add( t * alphas[i+4] * exp
  (roots_of_Q[i]*t), i=1..n_roots) ;
```

$$P\_over\_Q\_expanded\_power\_3\_integrated := -\frac{1}{2\left(s+\frac{1}{2}-\frac{1}{2}I\right)} - \frac{1}{2\left(s+\frac{1}{2}+\frac{1}{2}I\right)}$$

$$P\_over\_Q\_inverted\_power\_3 := -t\,e^{\left(-\frac{1}{2}+\frac{1}{2}I\right)t} - t\,e^{\left(-\frac{1}{2}-\frac{1}{2}I\right)t} \quad \textbf{(3.16)}$$

Expand complex number and simplify

```
> evalc(P_over_Q_inverted_power_1+P_over_Q_inverted_power_2+
  P_over_Q_inverted_power_3);
```

$$2\,t\,e^{-\frac{1}{2}t}\sin\left(\frac{1}{2}t\right) - 2\,t\,e^{-\frac{1}{2}t}\cos\left(\frac{1}{2}t\right) \quad \textbf{(3.17)}$$

```
> invlaplace( R, s, t) ;
```

$$\frac{1}{8}e^{-\frac{1}{2}t}\left(2\,t\sin\left(\frac{1}{2}t\right) - t^2\cos\left(\frac{1}{2}t\right)\right) \quad \textbf{(3.18)}$$