

Example of Constrained Minimization and KKT conditions

```
> restart;
```

```
> with(plots) :  
> with(LinearAlgebra) :
```

A function to minimize

```
> f := unapply(x^2*(1+sin(1-10*x)/10) + y^2 - exp(y), (x,y)) ;
```

$$f := (x, y) \rightarrow x^2 \left(1 - \frac{1}{10} \sin(-1 + 10x) \right) + y^2 - e^y \quad (1)$$

```
> g := unapply(1-x^2-2*y^2, (x,y)) ;
```

$$g := (x, y) \rightarrow 1 - x^2 - 2y^2 \quad (2)$$

The border of g1, i.e. where g1 = 0

```
> subs(x=cos(alpha), y=1/sqrt(2)*sin(alpha), g(x,y)) : simplify(%) ;
```

$$0 \quad (3)$$

```
> P1 := plot3d(f(x,y), x=-2..2, y=-2..2) ;
```

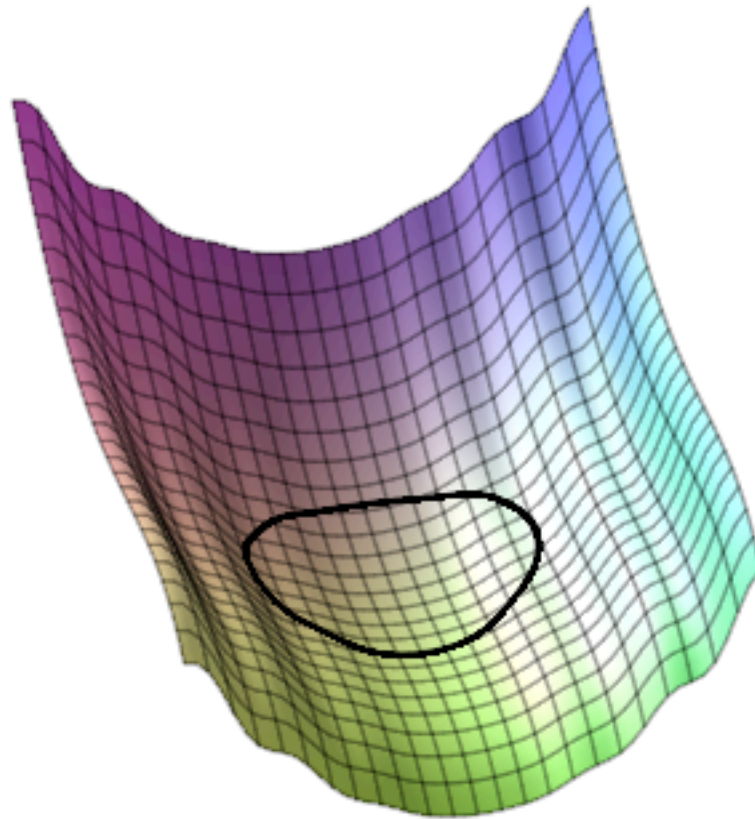
```
P2 := spacecurve([cos(t), sin(t)/sqrt(2), f(cos(t), sin(t)/sqrt(2))],  
t=0..2*Pi, thickness=3, color=black) ;
```

P1 := PLOT3D(...)

P2 := PLOT3D(...)

(4)

```
> display(P1, P2) ;
```



```
> with(Optimization):
> Minimize( f(x,y), {g(x,y)>=0} );
[-1.52811498164747217, [x = -2.32435496351070 10-14, y = 0.707106781186547]]
```

(5)

Build the nonlinear system using KKT conditions

```
> L := f(x,y) - mu*g(x,y);
L := x2 ( 1 - 1/10 sin(-1 + 10x) ) + y2 - ey - mu ( 1 - x2 - 2y2 )
```

(6)

```
> EQ1 := diff(L,x) ;
EQ2 := diff(L,y) ;
EQ3 := mu*g(x,y) ;
EQ1 := 2x ( 1 - 1/10 sin(-1 + 10x) ) - x2 cos(-1 + 10x) + 2mu x
EQ2 := 2y - ey + 4mu y
EQ3 := mu ( 1 - x2 - 2y2 )
```

(7)

```
> EQS := {EQ1 | (1..3)} ;
EQS := { mu ( 1 - x2 - 2y2 ), 2y - ey + 4mu y, 2x ( 1 - 1/10 sin(-1 + 10x) ) - x2 cos(-1
```

(8)

$$\left. + 10x) + 2\mu x \right\}$$

Solve numerically the KKT at first order

```
> SOL := fsolve( EQS, {x=0,y=0,mu=0} );
      SOL := {μ = 0.2170469283, x = 0., y = 0.7071067812} (9)
```

Compute the kernel of the gradients of the active constraints

```
> gradg := <diff(g(x,y),x) | diff(g(x,y),y)>;
      gradg1 := subs(SOL,gradg) ;
      gradg := [ -2 x -4 y ]
      gradg1 := [ -0. -2.828427125 ] (10)
```

```
> Z1 := op(NullSpace(gradg1)) ;
      Z1 := [ 1. ]
            [ 0. ] (11)
```

Compute the Hessian and check necessary conditions

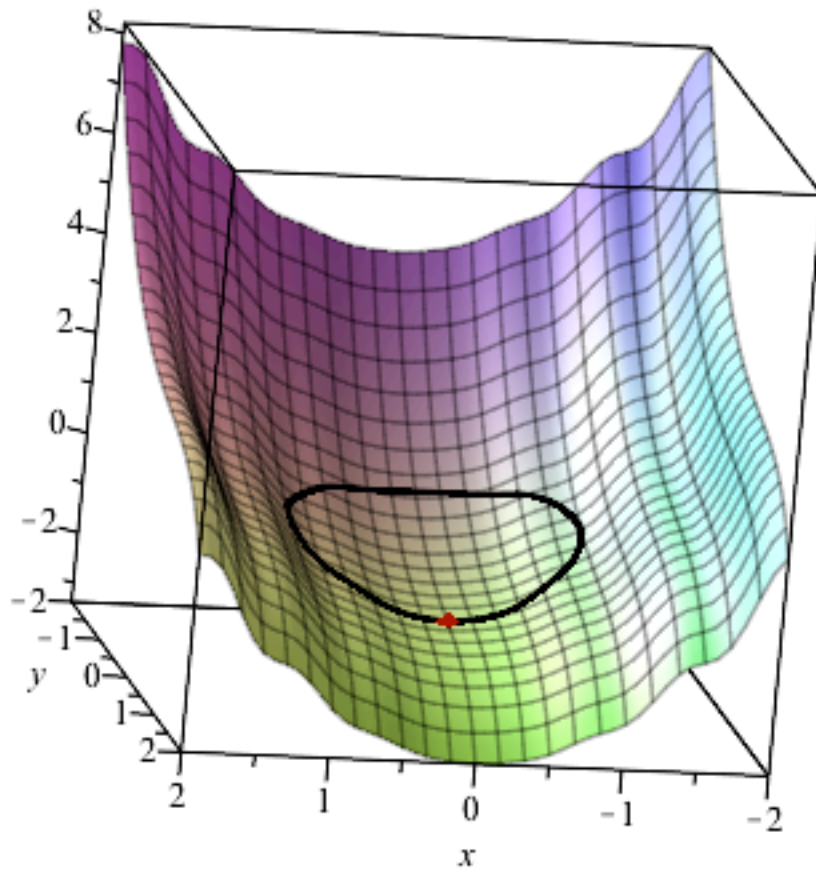
```
> hess := <<diff(f(x,y),x,x),diff(f(x,y),x,y)> | <diff(f(x,y),x,y),diff
      (f(x,y),y,y)>>;
      hess := [ 2 - 1/5 sin(-1 + 10x) - 4x cos(-1 + 10x) + 10x^2 sin(-1 + 10x) 0
                0 2 - e^y ] (12)
```

```
> hess1 := subs(SOL,hess) ;
      hess1 := [ 2.168294197 0
                 0 -0.028114982 ] (13)
```

```
> Transpose(Z1).hess1.Z1 ;
      2.16829419699999981 (14)
```

```
> P1 := plot3d(f(x,y),x=-2..2, y=-2..2) ;
      P2 := spacecurve( [cos(t),sin(t)/sqrt(2),f(cos(t),sin(t)/sqrt(2))],
      t=0..2*Pi,thickness=3,color=black) ;
      P3 := pointplot3d( subs(SOL,{[x,y,f(x,y)]}), symbolsize=20, color=
      red );
      P1 := PLOT3D(...)
      P2 := PLOT3D(...)
      P3 := PLOT3D(...) (15)
```

```
> display(P1,P2,P3) ;
```



```
> plot( f(cos(t), sin(t)/sqrt(2)), t=0..2*Pi) ;
```

