Example of Minimization of a Functional

```
> with(plots):
> a := 0 ;
  b := sqrt(3/2) ;
```

$$a := 0$$

$$b := \frac{1}{2}\sqrt{6} \qquad (1)$$

```
> J := Int( 2*y(x)^2+3*D(y)(x)^2,x=a..b) ;
```

$$J := \int_0^{\frac{1}{2}\sqrt{6}} \left(2\,y(x)^2 + 3\,\mathrm{D}(y)(x)^2\right) \mathrm{d}x \qquad (2)$$

```
> BC := y(a) = 1, y(b)=exp(1) ;
```

$$BC := y(0) = 1, y\left(\frac{1}{2}\sqrt{6}\right) = \mathrm{e} \qquad (3)$$

Solution using Euler-Lagrange equation

```
> F := op(1,J) ;
```

$$F := 2\,y(x)^2 + 3\,\mathrm{D}(y)(x)^2 \qquad (1.1)$$

```
> SUBS  := D(y)(x)=Dy, y(x)=y ;
```

$$SUBS := \mathrm{D}(y)(x) = Dy, y(x) = y \qquad (1.2)$$

```
> RSUBS := y=y(x), Dy = D(y)(x) ;
```

$$RSUBS := y = y(x), Dy = \mathrm{D}(y)(x) \qquad (1.3)$$

```
> FF := subs( SUBS, F) ;
```

$$FF := 2\,y^2 + 3\,Dy^2 \qquad (1.4)$$

Compute the Euler Lagrange equation

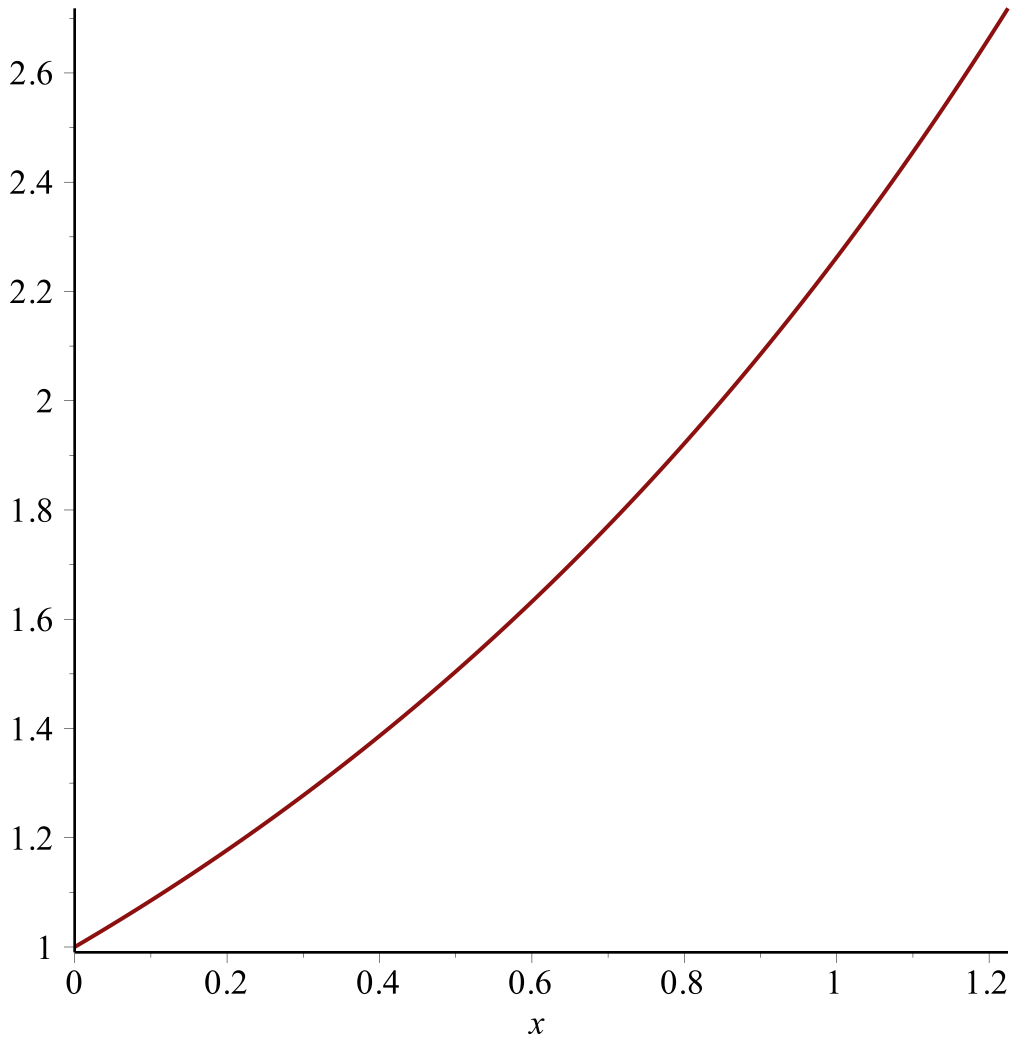
```
> ODE := subs( RSUBS, diff(FF,y) ) - diff( subs(RSUBS,diff(FF,Dy)),
  x ) ;
```

$$ODE := 4\,y(x) - 6\,\mathrm{D}^{(2)}(y)(x) \qquad (1.5)$$

```
> SOL := dsolve( {ODE,BC} ) ;
```

$$SOL := y(x) = \mathrm{e}^{\frac{1}{3}\sqrt{6}\,x} \qquad (1.6)$$

```
> plot( subs(SOL,y(x)), x=a..b ) ;
```



Numerical solution

```
> SUBSNUM := (D@@2)(y)(x)=(y(x+h)-2*y(x)+y(x-h))/h^2 ;
```

$$SUBSNUM := D^{(2)}(y)(x) = \frac{y(x+h) - 2\,y(x) + y(x-h)}{h^2} \quad \textbf{(1.7)}$$

```
> ODEN1 := subs( SUBSNUM, ODE ) ;
```

$$ODEN1 := 4\,y(x) - \frac{6\,(y(x+h) - 2\,y(x) + y(x-h))}{h^2} \quad \textbf{(1.8)}$$

Approximation by finite difference

```
> ODEFD := subs( y(x)=y[k], y(x+h)=y[k+1], y(x-h)=y[k-1], ODEN1 ) ;
```

$$ODEFD := 4\,y_k - \frac{6\,\left(y_{k+1} - 2\,y_k + y_{k-1}\right)}{h^2} \quad \textbf{(1.9)}$$

fix h and eval the Finite difference for k=1 up to N-1

```
> N := 10 ;
  h := (b-a)/N ;
```

$$N := 10$$

$$h := \frac{1}{20}\sqrt{6} \quad \textbf{(1.10)}$$

The linear system

```
> EQS := {seq(h^2*ODEFD,k=1..N-1)} union
         {op(subs(y(a)=y[0],y(b)=y[N],{BC})) } :
> NSOL := fsolve( EQS, {seq(y[k],k=0..N)} ) ;
```

$$NSOL := \{y_0 = 1., y_1 = 1.105221343, y_2 = 1.221494900, y_3 = 1.349983406, y_4 = 1.491971746, y_5 = 1.648879802, y_6 = 1.822276657, y_7 = 2.013896278, y_8 = 2.225654862, y_9 = 2.459669995, y_{10} = 2.718281828\} \quad \textbf{(1.11)}$$

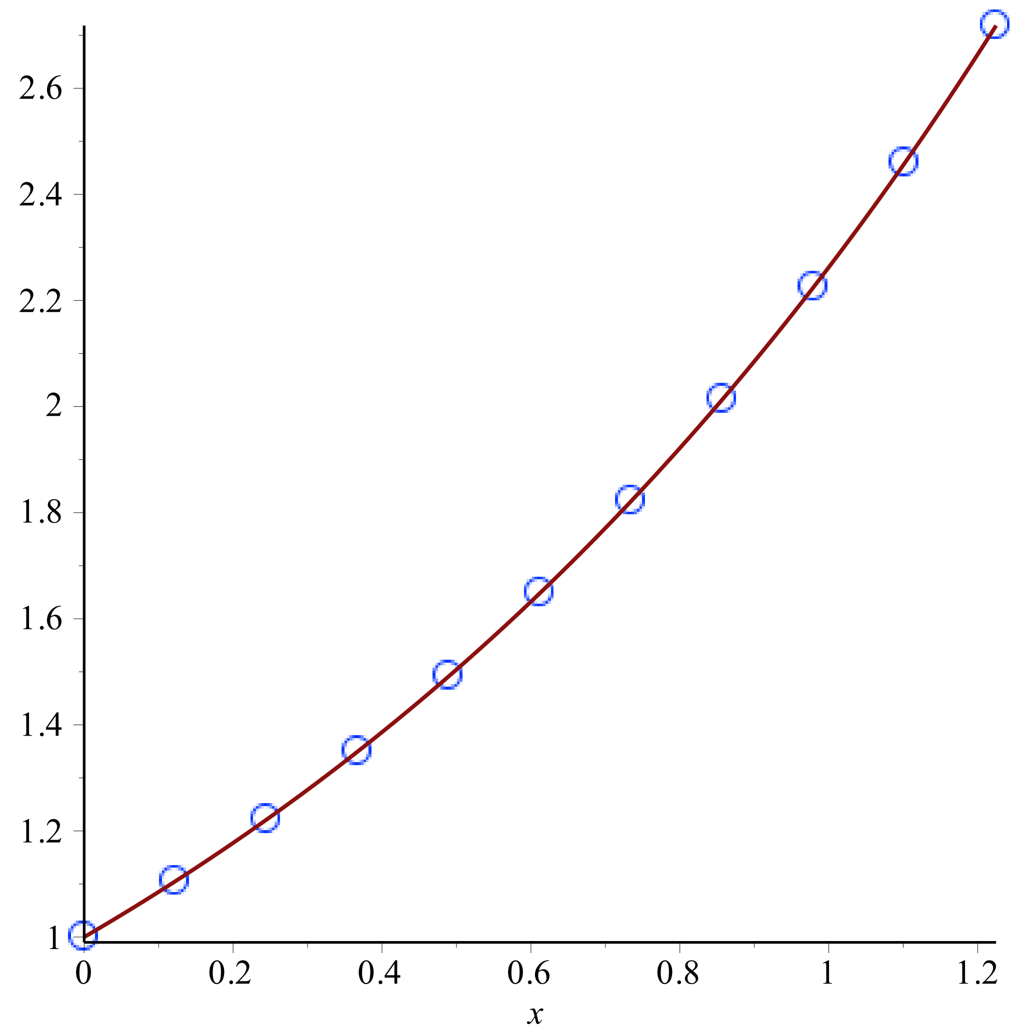
```
> XY := [seq( subs(NSOL,[evalf(a+k*h),y[k]]), k=0..N)] ;
```

$$XY := [[0., 1.], [0.1224744872, 1.105221343], [0.2449489743, 1.221494900], [0.3674234614, 1.349983406], [0.4898979486, 1.491971746], [0.6123724358, 1.648879802], [0.7348469229, 1.822276657], [0.8573214100, 2.013896278], [0.9797958972, 2.225654862], [1.102270384, 2.459669995], [1.224744872, 2.718281828]] \quad \textbf{(1.12)}$$

```
> A := plot(XY,color="blue",style=point, symbol=circle, symbolsize=
  20) ;
  B := plot( subs(SOL,y(x)), x=a..b ) ;
  display(A,B);
```

$$A := PLOT(\ldots)$$

$$B := PLOT(\ldots)$$



Solution using Direct Method (da finire)

fix h and eval the Finite difference for k=1 up to N-1

```
> N := 10 ;
  h := (b-a)/N ;
```

$$N := 10$$

$$h := \frac{1}{20}\sqrt{6} \quad \textbf{(2.1)}$$

Approximate the integral with a quadrature formula

```
> F := op(1,J) ;
```

$$F := 2\,y(x)^2 + 3\,\mathrm{D}(y)(x)^2 \quad \textbf{(2.2)}$$

```
> SUBS  := D(y)(x)=Dy, y(x)=y ;
```

$$SUBS := \mathrm{D}(y)(x) = Dy, y(x) = y \quad \textbf{(2.3)}$$

```
> RSUBS := y=y(x), Dy = D(y)(x) ;
```

$$RSUBS := y = y(x), Dy = \mathrm{D}(y)(x) \quad \textbf{(2.4)}$$

```
> FF := subs( SUBS, F) ;
```

$$FF := 2\,y^2 + 3\,Dy^2 \quad \textbf{(2.5)}$$

Compute the Euler Lagrange equation

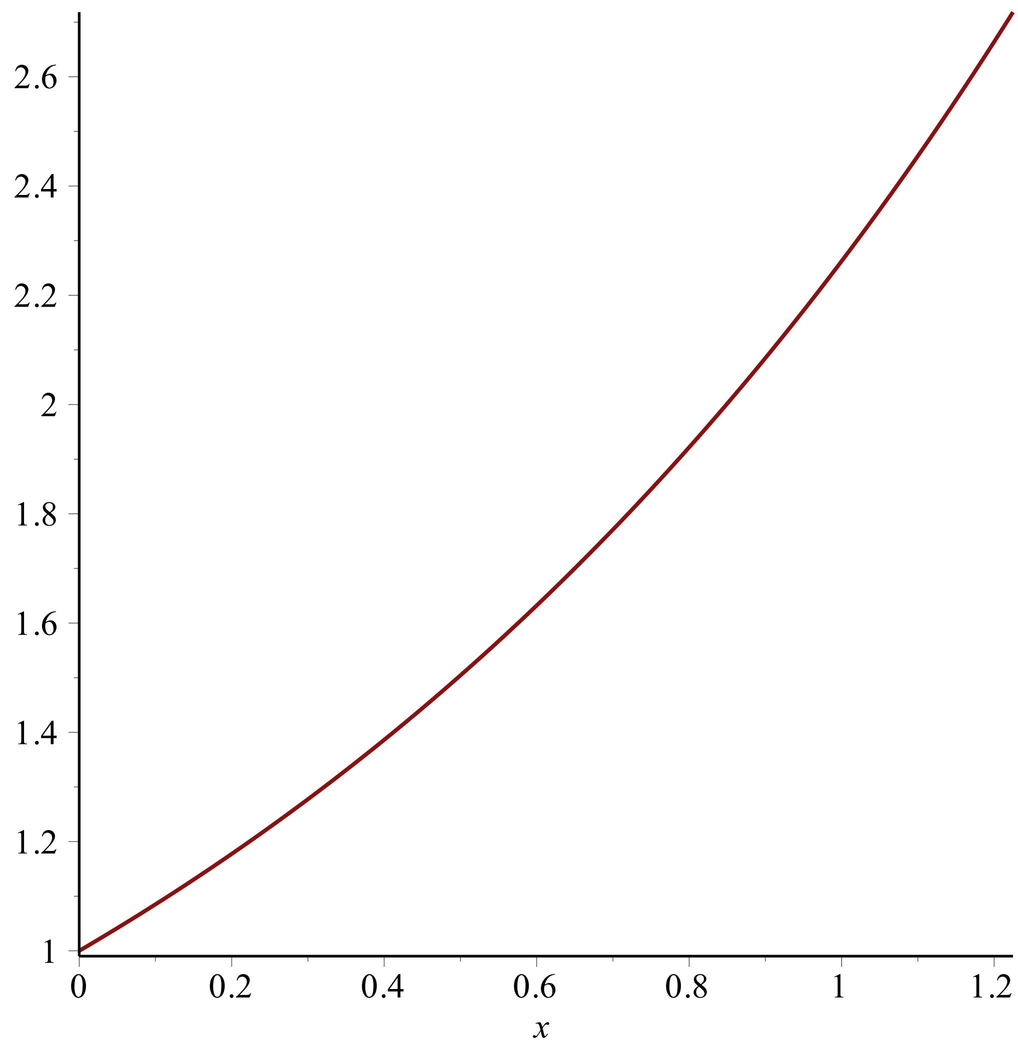
```
> ODE := subs( RSUBS, diff(FF,y) ) - diff( subs(RSUBS,diff(FF,Dy)),
  x ) ;
```

$$ODE := 4\,y(x) - 6\,D^{(2)}(y)(x) \quad \textbf{(2.6)}$$

```
> SOL := dsolve( {ODE,BC} ) ;
```

$$SOL := y(x) = e^{\frac{1}{3}\sqrt{6}\,x} \quad \textbf{(2.7)}$$

```
> plot( subs(SOL,y(x)), x=a..b ) ;
```



Numerical solution

```
> SUBSNUM := (D@@2)(y)(x)=(y(x+h)-2*y(x)+y(x-h))/h^2 ;
```

$$SUBSNUM := D^{(2)}(y)(x) = \frac{200}{3}\,y\left(x + \frac{1}{20}\sqrt{6}\right) - \frac{400}{3}\,y(x) + \frac{200}{3}\,y\left(x - \frac{1}{20}\sqrt{6}\right) \quad \textbf{(2.8)}$$

```
> ODEN1 := subs( SUBSNUM, ODE ) ;
```

$$ODEN1 := 804\,y(x) - 400\,y\left(x + \frac{1}{20}\sqrt{6}\right) - 400\,y\left(x - \frac{1}{20}\sqrt{6}\right) \quad \textbf{(2.9)}$$

Approximation by finite difference

```
> ODEFD := subs( y(x)=y[k], y(x+h)=y[k+1], y(x-h)=y[k-1], ODEN1 ) ;
```

$$ODEFD := 804\, y_k - 400\, y_{k+1} - 400\, y_{k-1} \quad (2.10)$$

fix h and eval the Finite difference for k=1 up to N-1

```
> N := 10 ;
  h := (b-a)/N ;
```

$$N := 10$$

$$h := \frac{1}{20}\sqrt{6} \quad (2.11)$$

The linear system

```
> EQS := {seq(h^2*ODEFD,k=1..N-1)} union
         {op(subs(y(a)=y[0],y(b)=y[N],{BC}))} :
> NSOL := fsolve( EQS, {seq(y[k],k=0..N)} ) ;
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$$NSOL := \{y_0 = 1., y_1 = 1.105221343, y_2 = 1.221494900, y_3 = 1.349983406, y_4 = 1.491971746, y_5 = 1.648879802, y_6 = 1.822276657, y_7 = 2.013896278, y_8 = 2.225654862, y_9 = 2.459669995, y_{10} = 2.718281828\} \quad (2.12)$$

```
> XY := [seq( subs(NSOL,[evalf(a+k*h),y[k]]), k=0..N)] ;
```

$$XY := [[0., 1.], [0.1224744872, 1.105221343], [0.2449489743, 1.221494900], [0.3674234614, 1.349983406], [0.4898979486, 1.491971746], [0.6123724358, 1.648879802], [0.7348469229, 1.822276657], [0.8573214100, 2.013896278], [0.9797958972, 2.225654862], [1.102270384, 2.459669995], [1.224744872, 2.718281828]] \quad (2.13)$$

```
> A := plot(XY,color="blue",style=point, symbol=circle, symbolsize=
  20) ;
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  display(A,B);
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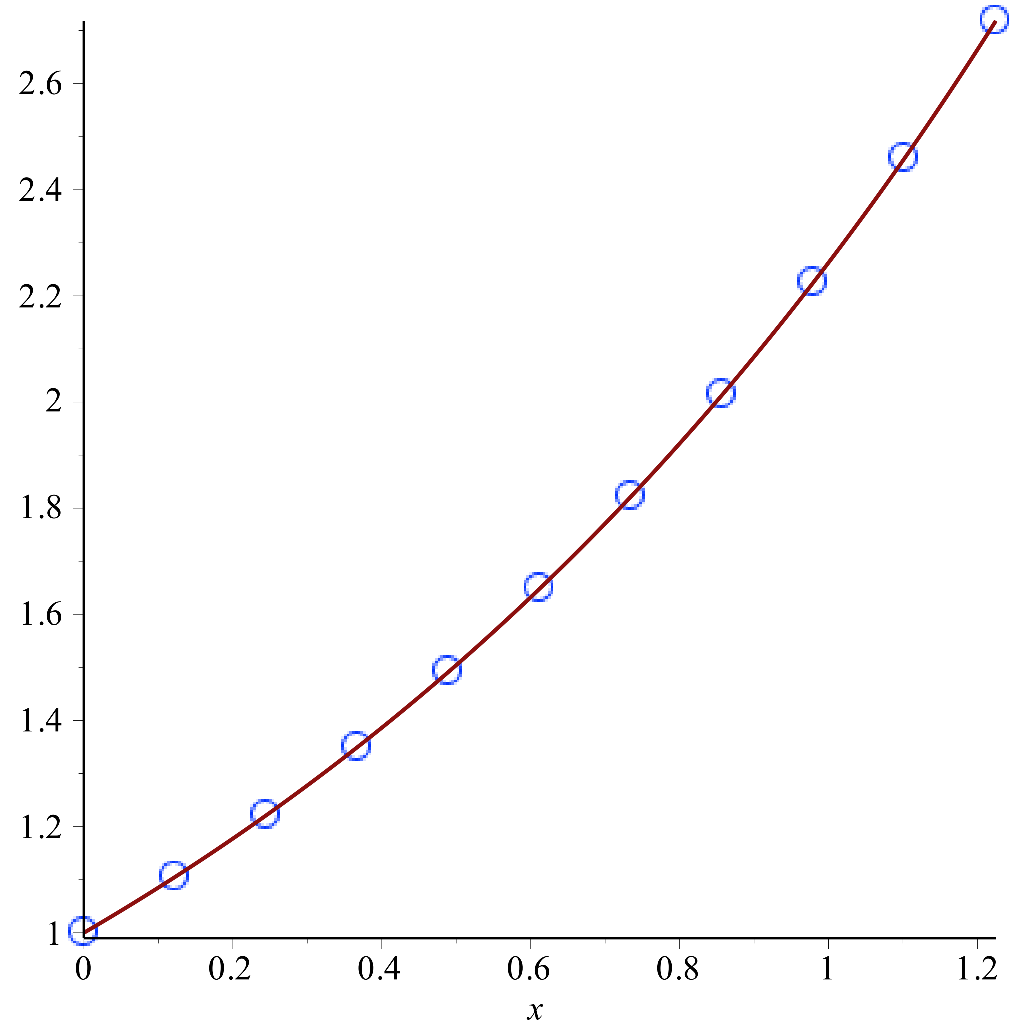
$$A := PLOT(...)$$

$$B := PLOT(...)$$


2.6
2.4
2.2
2
1.8
1.6
1.4
1.2
1
0
0.2
0.4
0.6
0.8
1
1.2
x