

Connection between Laplace transform and Bode plot

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A simple example

Consider the following ODE

$$x''(t) + 3x'(t) + 2x(t) = \cos(t), \quad (1)$$

which has the general solution:

$$x(t) = \underbrace{(x(0) + x'(0)) (2e^{-t} - e^{-2t}) + \frac{2}{5}e^{-2t} - \frac{1}{2}e^{-t}}_{\text{goes } \rightarrow 0 \text{ as } t \rightarrow \infty} + \frac{1}{10} \cos(t) + \frac{3}{10} \sin(t)$$

so that we can write

$$x(t) \approx \frac{1}{10} \cos(t) + \frac{3}{10} \sin(t), \quad \text{for } t \text{ large}$$

from the identity $\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$ we can deduce

$$x(t) \approx \frac{1}{10} \cos(t) + \frac{3}{10} \sin(t) = \frac{1}{\sqrt{10}} \cos(t - \phi) \quad \text{for } t \text{ large}$$

where $\phi = \arctan(3)$. Thus we considering $\sin(t)$ as the input of the ODE (1) and $x(t)$ as the output we can say that *asymptotically* the input $\sin(t)$ reduce its amplitude by a factor $1/\sqrt{10}$ and is shifted backward by the angle ϕ . Considering now a *generic* frequency $\cos(\omega t)$ as input we have the ODE

$$x''(t) + 3x'(t) + 2x(t) = \cos(\omega t), \quad (2)$$

which has the general solution

$$x(t) = \underbrace{A(\omega)e^{-t} + B(\omega)e^{-2t}}_{\text{goes } \rightarrow 0 \text{ as } t \rightarrow \infty} + \frac{(2 - \omega^2) \cos(\omega t) + 3\omega \sin(\omega t)}{(4 + \omega^2)(1 + \omega^2)}$$

where

$$A(\omega) = \frac{(2\omega^4 + 10\omega^2 + 8)x_0 + (\omega^4 + 5\omega^2 + 4)x'_0 - (4 + \omega^2)}{(4 + \omega^2)(1 + \omega^2)},$$

$$B(\omega) = \frac{2(\omega^2 + 1) - (\omega^4 + 5\omega^2 + 4)(x_0 + x'_0)}{(4 + \omega^2)(1 + \omega^2)}$$

as for the ODE (1) we can write

$$x(t) \approx \frac{(2 - \omega^2) \cos(\omega t) + 3\omega \sin(\omega t)}{(4 + \omega^2)(1 + \omega^2)}, \quad \text{for } t \text{ large}$$

from the identity $\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$ we can deduce

$$x(t) \approx \frac{(2 - \omega^2) \cos(\omega t) + 3\omega \sin(\omega t)}{(4 + \omega^2)(1 + \omega^2)} = \frac{\cos(\omega t - \phi)}{\sqrt{(4 + \omega^2)(1 + \omega^2)}} \quad \text{for } t \text{ large}$$

where $\phi = \arctan(3\omega/(2 - \omega^2))$. Thus we considering $\sin(t)$ as the input of the ODE (2) and $x(t)$ as the output we can say that *asymptotically* the input $\sin(t)$ reduce its amplitude by a factor $1/\sqrt{(4 + \omega^2)(1 + \omega^2)}$ and is shifted backward by the angle ϕ .

Generalization

Consider now a generic linear ODE

$$x^{(N)} + \sum_{k=0}^{N-1} a_k x^{(k)}(t) = \cos(\omega t), \quad x(0) = x'(0) = \dots = x^{(N-1)}(0) = 0, \quad (3)$$

the response $x(t)$ to the sinusoidal input $\cos(\omega t)$ is in general

$$x(t) = x_0(t) + A \cos(\omega t + \phi),$$

where $x_0(t) \rightarrow 0$ if $t \rightarrow \infty$ if the homogenous part of the ODE, i.e. $\sum_{k=0}^N a_k x^{(k)}(t) = 0$, is stable. In this case we say that the signal $\cos(\omega t)$ is gained or reduced by a factor A and shifted by an angle ϕ . If we take the Laplace transform of (3) we have

$$\left(s^N + \sum_{k=0}^{N-1} a_k s^k \right) x(s) = \frac{s}{s^2 + \omega^2},$$

and if s_0, s_1, \dots, s_{N-1} are the root of the polynomial $s^N + \sum_{k=0}^{N-1} a_k s^k$ we have

$$\prod_{k=0}^{N-1} (s - s^k) x(s) = \frac{s}{s^2 + \omega^2},$$

and thus,

$$x(s) = G(s) \times \frac{s}{s^2 + \omega^2} \quad (4)$$

where

$$G(s) = \frac{1}{\prod_{k=0}^{N-1} (s - s^k)}$$

is the transfer function of the ODE. Using simple fraction expansion and for simplicity assuming all the root simple we have

$$x(s) = \frac{A(\omega)s - B(\omega)\omega}{s^2 + \omega^2} + \sum_{k=0}^N \frac{C_k}{s - s^k} \quad (5)$$

where $A(\omega)$, $B(\omega)$ and C_k are computed by equating (4) with (5). Reversing Laplace transform from $x(s)$ we have (remember $\Re(s_k) < 0$ because ODE is stable):

$$x(t) = K(\omega) \cos(\omega t + \phi(\omega)) + \underbrace{\sum_{k=0}^N C_k e^{ts^k}}_{\text{goes } \rightarrow 0 \text{ as } t \rightarrow \infty}$$

where

$$K(\omega) = \sqrt{A(\omega)^2 + B(\omega)^2}, \quad \phi(\omega) = \arctan\left(\frac{B(\omega)}{A(\omega)}\right). \quad (6)$$

To compute $K(\omega)$ and $\phi(\omega)$ we need to compute $A(\omega)$ and $B(\omega)$. To this purpose from (4) and (5) we have

$$G(s) \times \frac{s}{s^2 + \omega^2} = \frac{A(\omega)s - B(\omega)\omega}{s^2 + \omega^2} + \sum_{k=0}^N \frac{C_k}{s - s^k}$$

multiply both side by $s - i\omega$ we have

$$G(s) \times \frac{s}{s + i\omega} = \frac{A(\omega)s - B(\omega)\omega}{s + i\omega} + (s - i\omega) \sum_{k=0}^N \frac{C_k}{s - s^k}$$

and computing equation in $s = i\omega$ we have

$$G(i\omega) \times \frac{1}{2} = \frac{A(\omega)i\omega - B(\omega)\omega}{2i\omega} + 0 \times \sum_{k=0}^N \frac{C_k}{s - s^k}$$

and thus

$$G(i\omega) = A(\omega) - \frac{B(\omega)}{i} = A(\omega) + iB(\omega)$$

and from (6) we have

$$K(\omega) = \sqrt{A(\omega)^2 + B(\omega)^2} = |G(i\omega)|,$$

$$\phi(\omega) = \arctan\left(\frac{B(\omega)}{A(\omega)}\right) = \arctan\left(\frac{\Im(G(i\omega))}{\Re(G(i\omega))}\right).$$