# Pendulum in cartesian coordinates

# Taylor based numerical scheme with Baumgarte stabilization

```
> restart:
```

Pendulum equation

```
> EQ1 := mass*diff(x(t),t,t)+2*x(t)*lambda(t) ;
  EQ2 := mass*diff(y(t),t,t)+2*y(t)*lambda(t)+mass*g ;
  EQ3 := x(t)^2+y(t)^2-1 ;
```

$$EQ1 := mass\left(\frac{d^2}{dt^2}x(t)\right)+2\,x(t)\,\lambda(t)$$

$$EQ2 := mass\left(\frac{d^2}{dt^2}y(t)\right)+2\,y(t)\,\lambda(t)+mass\,g$$

$$EQ3 := x(t)^2+y(t)^2-1 \tag{1}$$

Derivate constraint two times

```
> DEQ3  := diff(EQ3,t);
  DDEQ3 := diff(DEQ3,t);
```

$$DEQ3 := 2\,x(t)\left(\frac{d}{dt}x(t)\right)+2\,y(t)\left(\frac{d}{dt}y(t)\right)$$

$$DDEQ3 := 2\left(\frac{d}{dt}x(t)\right)^2+2\,x(t)\left(\frac{d^2}{dt^2}x(t)\right)+2\left(\frac{d}{dt}y(t)\right)^2+2\,y(t)\left(\frac{d^2}{dt^2}y(t)\right) \tag{2}$$

Substitute DDEQ3 with stabilized equation

```
> SEQ3 := DDEQ3 + 2*zeta*omega*DEQ3 + omega^2*EQ3
Warning, inserted missing semicolon at end of statement
```

$$SEQ3 := 2\left(\frac{d}{dt}x(t)\right)^2+2\,x(t)\left(\frac{d^2}{dt^2}x(t)\right)+2\left(\frac{d}{dt}y(t)\right)^2+2\,y(t)\left(\frac{d^2}{dt^2}y(t)\right)+2\,\zeta\,\omega\left(2\,x(t)\left(\frac{d}{dt}x(t)\right)+2\,y(t)\left(\frac{d}{dt}y(t)\right)\right)+\omega^2\left(x(t)^2+y(t)^2-1\right) \tag{3}$$

Solve for second derivative

```
> RESACC := solve( {EQ1,EQ2}, diff({x(t),y(t)},t,t) ) ;
```

$$RESACC := \left\{\frac{d^2}{dt^2}x(t)=-\frac{2\,x(t)\,\lambda(t)}{mass},\ \frac{d^2}{dt^2}y(t)=-\frac{2\,y(t)\,\lambda(t)+mass\,g}{mass}\right\} \tag{4}$$

Solve for multiplier

```
> RESLAMBDA := solve( subs(RESACC,SEQ3), {lambda(t)} ) ;
```

$$RESLAMBDA := \left\{\lambda(t)=-\frac{1}{4}\,\frac{1}{x(t)^2+y(t)^2}\left(mass\left(-2\left(\frac{d}{dt}x(t)\right)^2-2\left(\frac{d}{dt}y(t)\right)^2+2\,y(t)\,g-4\,\zeta\,\omega\,x(t)\left(\frac{d}{dt}x(t)\right)-4\,\zeta\,\omega\,y(t)\left(\frac{d}{dt}y(t)\right)-\omega^2\,x(t)^2-\omega^2\,y(t)^2\right.\right.\right. \tag{5}$$

$$+ \omega^2 \Big) \Big) \Big\}$$

Change names

```
> SUBS := { diff(x(t),t,t) = ax(t),
            diff(y(t),t,t) = ay(t),
            diff(x(t),t)   = u(t),
            diff(y(t),t)   = v(t) } ;
```

$$SUBS := \left\{ \frac{d}{dt} x(t) = u(t),\ \frac{d}{dt} y(t) = v(t),\ \frac{d^2}{dt^2} x(t) = ax(t),\ \frac{d^2}{dt^2} y(t) = ay(t) \right\} \quad \textbf{(6)}$$

```
> subs( SUBS, RESLAMBDA ) ;
```

$$\Big\{ \lambda(t) = -\frac{1}{4} \frac{1}{x(t)^2 + y(t)^2} \Big( mass \Big( -2\, u(t)^2 - 2\, v(t)^2 + 2\, y(t)\, g - 4\, \zeta\, \omega\, x(t)\, u(t) - 4\, \zeta\, \omega\, y(t)\, v(t) - \omega^2\, x(t)^2 - \omega^2\, y(t)^2 + \omega^2 \Big) \Big) \Big\} \quad \textbf{(7)}$$

Advancing with Taylor

```
> XKP1 := x(t)+u(t)*DT+ax(t)*DT^2/2 ;
  YKP1 := y(t)+v(t)*DT+ay(t)*DT^2/2 ;
  UKP1 := u(t)+ax(t)*DT ;
  VKP1 := v(t)+ay(t)*DT ;
```

$$XKP1 := x(t) + u(t)\, DT + \frac{1}{2}\, ax(t)\, DT^2$$

$$YKP1 := y(t) + v(t)\, DT + \frac{1}{2}\, ay(t)\, DT^2$$

$$UKP1 := u(t) + ax(t)\, DT$$

$$VKP1 := v(t) + ay(t)\, DT \quad \textbf{(8)}$$

Substituting acceleration

```
> XKP1 := subs( subs(SUBS,RESACC), XKP1) ;
  YKP1 := subs( subs(SUBS,RESACC), YKP1) ;
  UKP1 := subs( subs(SUBS,RESACC), UKP1) ;
  VKP1 := subs( subs(SUBS,RESACC), VKP1) ;
```

$$XKP1 := x(t) + u(t)\, DT - \frac{x(t)\, \lambda(t)\, DT^2}{mass}$$

$$YKP1 := y(t) + v(t)\, DT - \frac{1}{2} \frac{\left(2\, y(t)\, \lambda(t) + mass\, g\right) DT^2}{mass}$$

$$UKP1 := u(t) - \frac{2\, x(t)\, \lambda(t)\, DT}{mass}$$

$$VKP1 := v(t) - \frac{\left(2\, y(t)\, \lambda(t) + mass\, g\right) DT}{mass} \quad \textbf{(9)}$$

Substituting Lambda

```
> XKP1 := subs( subs(SUBS,RESLAMBDA), XKP1) ;
  YKP1 := subs( subs(SUBS,RESLAMBDA), YKP1) ;
  UKP1 := subs( subs(SUBS,RESLAMBDA), UKP1) ;
  VKP1 := subs( subs(SUBS,RESLAMBDA), VKP1) ;
```

$$XKP1 := x(t) + u(t)\, DT + \frac{1}{4} \frac{1}{x(t)^2 + y(t)^2} \Big(x(t) \big(-2\, u(t)^2 - 2\, v(t)^2 + 2\, y(t)\, g - 4\, \zeta\, \omega\, x(t)\, u(t) - 4\, \zeta\, \omega\, y(t)\, v(t) - \omega^2\, x(t)^2 - \omega^2\, y(t)^2 + \omega^2\big)\, DT^2\Big)$$

$$YKP1 := y(t) + v(t)\, DT - \frac{1}{2} \frac{1}{mass} \left(\left(-\frac{1}{2} \frac{1}{x(t)^2 + y(t)^2} \Big(y(t)\, mass \big(-2\, u(t)^2 - 2\, v(t)^2 + 2\, y(t)\, g - 4\, \zeta\, \omega\, x(t)\, u(t) - 4\, \zeta\, \omega\, y(t)\, v(t) - \omega^2\, x(t)^2 - \omega^2\, y(t)^2 + \omega^2\big)\Big) + mass\, g\right) DT^2\right)$$

$$UKP1 := u(t) + \frac{1}{2} \frac{1}{x(t)^2 + y(t)^2} \Big(x(t) \big(-2\, u(t)^2 - 2\, v(t)^2 + 2\, y(t)\, g - 4\, \zeta\, \omega\, x(t)\, u(t) - 4\, \zeta\, \omega\, y(t)\, v(t) - \omega^2\, x(t)^2 - \omega^2\, y(t)^2 + \omega^2\big)\, DT\Big)$$

$$VKP1 := v(t) - \frac{1}{mass} \left(\left(-\frac{1}{2} \frac{1}{x(t)^2 + y(t)^2} \Big(y(t)\, mass \big(-2\, u(t)^2 - 2\, v(t)^2 + 2\, y(t)\, g - 4\, \zeta\, \omega\, x(t)\, u(t) - 4\, \zeta\, \omega\, y(t)\, v(t) - \omega^2\, x(t)^2 - \omega^2\, y(t)^2 + \omega^2\big)\Big) + mass\, g\right) DT\right) \quad \textbf{(10)}$$

Build numerical scheme

```
> SUBSV := { x(t)=x0, y(t)=y0, u(t)=u0, v(t)=v0, mu(t)=muN } ;
```

$$SUBSV := \{x(t) = x0, y(t) = y0, u(t) = u0, v(t) = v0, \mu(t) = muN\} \quad \textbf{(11)}$$

```
> XKP1 := subs(SUBSV,XKP1) ;
  YKP1 := subs(SUBSV,YKP1) ;
  UKP1 := subs(SUBSV,UKP1) ;
  VKP1 := subs(SUBSV,VKP1) ;
```

$$XKP1 := x0 + u0\, DT + \frac{1}{4} \frac{1}{x0^2 + y0^2} \Big(x0 \big(-2\, u0^2 - 2\, v0^2 + 2\, y0\, g - 4\, \zeta\, \omega\, x0\, u0 - 4\, \zeta\, \omega\, y0\, v0 - \omega^2\, x0^2 - \omega^2\, y0^2 + \omega^2\big)\, DT^2\Big)$$

$$YKP1 := y0 + v0\, DT - \frac{1}{2} \frac{1}{mass} \left(\left(-\frac{1}{2} \frac{1}{x0^2 + y0^2} \Big(y0\, mass \big(-2\, u0^2 - 2\, v0^2 + 2\, y0\, g - 4\, \zeta\, \omega\, x0\, u0 - 4\, \zeta\, \omega\, y0\, v0 - \omega^2\, x0^2 - \omega^2\, y0^2 + \omega^2\big)\Big) + mass\, g\right) DT^2\right)$$

$$UKP1 := u0 + \frac{1}{2} \frac{1}{x0^2 + y0^2} \Big(x0 \big(-2\, u0^2 - 2\, v0^2 + 2\, y0\, g - 4\, \zeta\, \omega\, x0\, u0 - 4\, \zeta\, \omega\, y0\, v0 - \omega^2\, x0^2 - \omega^2\, y0^2 + \omega^2\big)\, DT\Big)$$

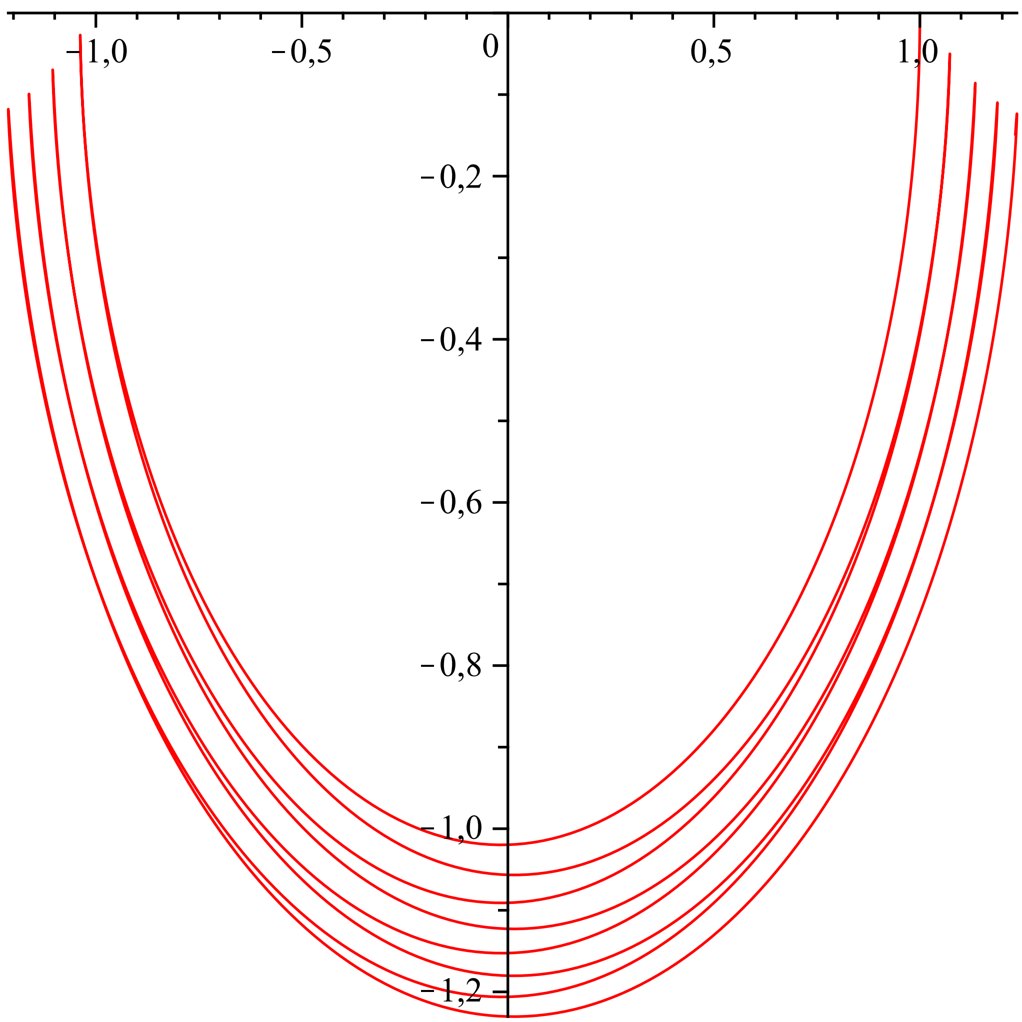
$$VKP1 := v0 - \frac{1}{mass} \left(\left(-\frac{1}{2} \frac{1}{x0^2 + y0^2} \Big(y0\, mass \big(-2\, u0^2 - 2\, v0^2 + 2\, y0\, g - 4\, \zeta\, \omega\, x0\, u0 - 4\, \zeta\, \omega\, y0\, v0 - \omega^2\, x0^2 - \omega^2\, y0^2 + \omega^2\big)\Big) + mass\, g\right) DT\right) \quad \textbf{(12)}$$

```
> advance := proc ( x0, y0, u0, v0, dt, N )
    local kk, SUBS, x1, y1, u1, v1, XY,UV ;
```

```
    XY := [[x0,y0]] ;
    UV := [[u0,v0]] ;
    for kk from 1 to N do
      SUBS := { g=9.81, DT=dt, omega=0.01, zeta=1,
                xO=XY[-1][1], yO=XY[-1][2],
                uO=UV[-1][1], vO=UV[-1][2]} ;
      x1  := evalf(subs( SUBS, XKP1 )) ;
      y1  := evalf(subs( SUBS, YKP1 )) ;
      u1  := evalf(subs( SUBS, UKP1 )) ;
      v1  := evalf(subs( SUBS, VKP1 )) ;
      XY  := [op(XY),[x1,y1]] ;
      UV  := [op(UV),[u1,v1]] ;
    end  ;
    [XY,UV] ;
  end proc:
```

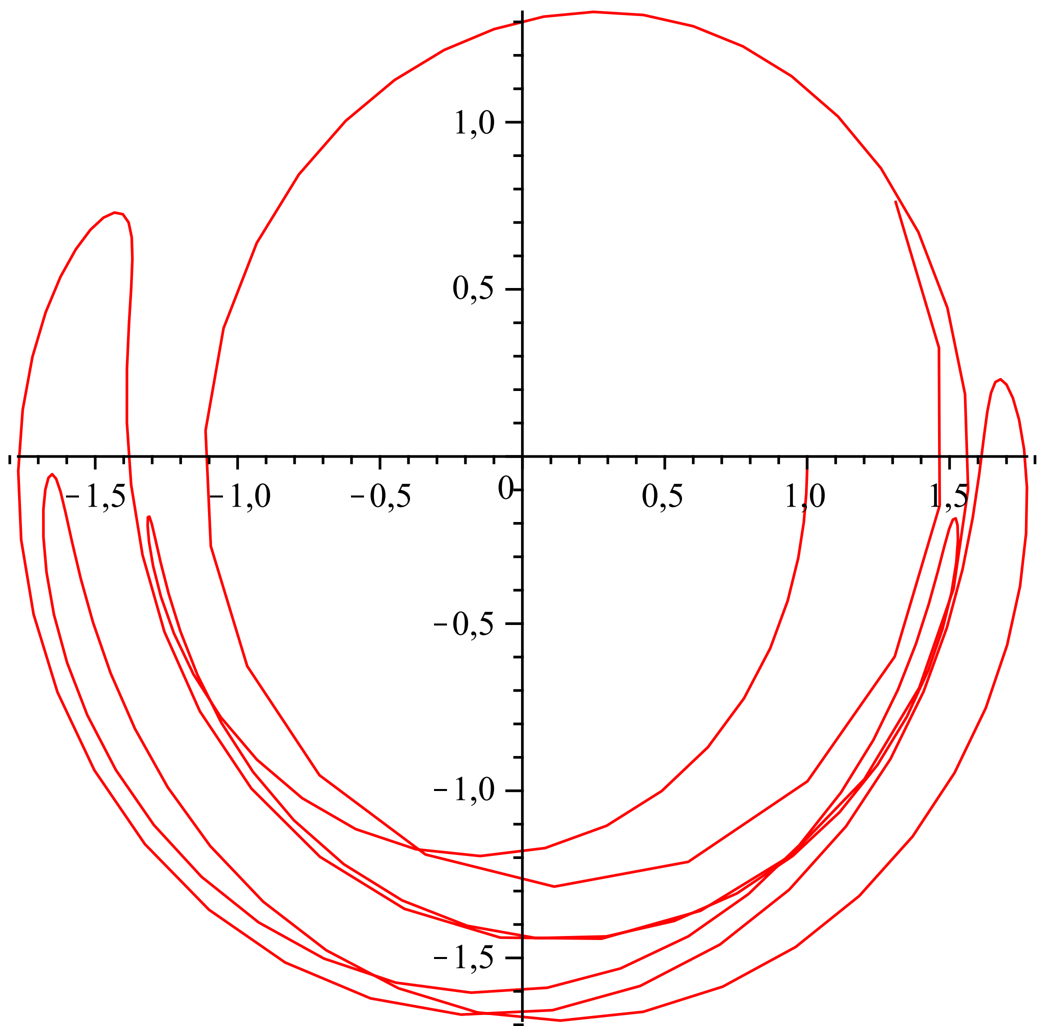
Test numerical scheme DT = 1/200

```
> RES := advance( 1, 0, 0, 0, 10/2000, 2000 ) :
> plot( RES[1] ) ;
```




Test numerical scheme DT = 1/20

```
> RES := advance( 1, 0, 0, 0, 10/200, 200 ) :
  plot( RES[1] ) ;
```


1,0
0,5
−1,5
−1,0
−0,5
0
0,5
1,0
1,5
−0,5
−1,0
−1,5

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