Tables and summary

"Numerical Methods for Dynamic System and Control"

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Fourier Serie

$$S_f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty}\left(a_k \cos\frac{k\pi x}{\ell} + b_k \sin\frac{k\pi x}{\ell}\right),$$

$$\begin{aligned} a_0 &= \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\,\mathrm{d}x, \\ a_k &= \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\cos\frac{k\pi x}{\ell}\,\mathrm{d}x, \\ b_k &= \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\sin\frac{k\pi x}{\ell}\,\mathrm{d}x. \end{aligned}$$

Z and Laplace transform

Z TRANSFORM TABLE: $\sum_{k=0}^{\infty} f_k z^{-k}$	
$k_\ell = k(k-1)\cdots(k-\ell+1) = \dfrac{k!}{(k-\ell)!}$	
δ_k	1
$\mathbf{1}_k$	$\dfrac{z}{z-1}$
a^k	$\dfrac{z}{z-a}$

$k\,a^k$	$\dfrac{za}{(z-a)^2}$
$k(k-1)\,a^k$	$\dfrac{2za^2}{(z-a)^3}$
$k_\ell\,a^k$	$\ell!\dfrac{za^\ell}{(z-a)^{\ell+1}}$
$a^k\binom{k}{\ell}$	$\dfrac{z\,a^\ell}{(z-a)^{\ell+1}}$
$a^k\,f_k$	$\widetilde{f}\left(\dfrac{z}{a}\right)$
$k\,f_k$	$-z\dfrac{\mathrm{d}\widetilde{f}(z)}{\mathrm{d}z}$
$k^2\,f_k$	$\left(z\dfrac{\mathrm{d}}{\mathrm{d}z}\right)\left(z\dfrac{\mathrm{d}}{\mathrm{d}z}\right)\widetilde{f}(z)$
$k^\ell\,f_k$	$\left(-z\dfrac{\mathrm{d}}{\mathrm{d}z}\right)^\ell\widetilde{f}(z)$
$f_{k+\ell}$	$z^\ell\left(\widetilde{f}(z)-\sum_{j=0}^{\ell-1}f_j z^{-j}\right)$
$f_{k-\ell}$	$z^{-\ell}\widetilde{f}(z)$
$k_\ell\,f_{k-\ell}$	$(-1)^\ell z\dfrac{\mathrm{d}^\ell}{\mathrm{d}z^\ell}\left(\dfrac{1}{z}\widetilde{f}(z)\right)$
$(f\star g)_k$	$\widetilde{f}(z)\widetilde{g}(z)$
$a^k\sin\omega k$	$\dfrac{za\sin\omega}{z^2-2za\cos\omega+a^2}$
$\ldots$	$\dfrac{za\sin\omega}{(z-a\cos\omega)^2+(a\sin\omega)^2}$
$a^k\cos\omega k$	$\dfrac{z^2-za\cos\omega}{z^2-2za\cos\omega+a^2}$
$\ldots$	$\dfrac{z^2-za\cos\omega}{(z-a\cos\omega)^2+(a\sin\omega)^2}$

Laplace Transform Table $\int_0^\infty f(t)e^{-st}\,\mathrm{d}t$	
$a\,f(t) + b\,g(t)$	$a\,\widehat{f}(s) + b\,\widehat{g}(s)$
$f(at)$	$\frac{1}{a}\,\widehat{f}\left(\frac{s}{a}\right) \quad [a > 0]$
$e^{at} f(t)$	$\widehat{f}(s-a)$
$f(t-a)$	$e^{-as}\,\widehat{f}(s)$
$1, \quad t, \quad t^k$	$\frac{1}{s}, \quad \frac{1}{s^2}, \quad \frac{k!}{s^{k+1}}$
a^{bt}	$\frac{1}{s - b\log a}$
$\int_0^t f(z)\,\mathrm{d}z$	$\frac{1}{s}\widehat{f}(s)$
$f'(t)$	$s\widehat{f}(s) - f(0^+)$
$f''(t)$	$s^2\widehat{f}(s) - f'(0^+) - s f(0^+)$
$\frac{\mathrm{d}^n}{\mathrm{d}t^n} f(t)$	$s^n \widehat{f}(s) - \sum_{j=0}^{n-1} s^{n-j-1} f^{(j)}(0^+)$
$t^n f(t)$	$(-1)^n \frac{\mathrm{d}^n}{\mathrm{d}s^n}\widehat{f}(s)$
$(f \star g)(t)$	$\widehat{f}(s)\,\widehat{g}(s)$
$e^{at}\cos\omega t, \quad e^{at}\sin\omega t$	$\frac{s-a}{(s-a)^2+\omega^2}, \quad \frac{\omega}{(s-a)^2+\omega^2}$
$e^{at}\cosh\omega t, \quad e^{at}\sinh\omega t$	$\frac{s-a}{(s-a)^2-\omega^2}, \quad \frac{\omega}{(s-a)^2-\omega^2}$
$e^{at} t^n$	$\frac{n!}{(s-a)^{n+1}}$
$e^{\alpha t} - e^{\beta t}$	$\frac{\alpha - \beta}{(s-\alpha)(s-\beta)}$

Constrained minima and Lagrange multiplier

Consider the constrained minimization problem

$$\begin{aligned}&\text{minimize:} && f(\boldsymbol{x}) \\ &\text{subject to:} && h_i(\boldsymbol{x}) = 0 \qquad i = 1, 2, \ldots, m\end{aligned}$$

Solution algorithm

- Compute the Lagrangian function: $\mathcal{L}(\boldsymbol{x}, \boldsymbol{\lambda}) = f(\boldsymbol{x}) - \sum_{k=1}^{m} \lambda_k h_k(\boldsymbol{x})$
- Solve the nonlinear system $\nabla_x \mathcal{L}(\boldsymbol{x}, \boldsymbol{\lambda}) = \mathbf{0}^T$ with $\boldsymbol{h}(\boldsymbol{x}) = \mathbf{0}$.
- For each solution points $(\boldsymbol{x}^\star, \boldsymbol{\lambda}^\star)$ compute $\nabla \boldsymbol{h}(\boldsymbol{x}^\star)$ and check it is full rank, or the rows are linearly indiependent.
- Compute the matrix $\boldsymbol{K}$ the kernel of $\nabla \boldsymbol{h}(\boldsymbol{x}^\star)$, i.e. $\nabla \boldsymbol{h}(\boldsymbol{x}^\star)\boldsymbol{K} = \mathbf{0}$.
- Compute the reduce Hessian

$$\boldsymbol{H} = \boldsymbol{K}^T \nabla_x^2 \mathcal{L}(\boldsymbol{x}^\star, \boldsymbol{\lambda}^\star) \boldsymbol{K},$$

  - Necessary condition: $\boldsymbol{H}$ is semi-positive definite.
  - Sufficient condition: $\boldsymbol{H}$ is positive definite.

The following theorem prove the sufficient condition.

Theorem 1 (of Lagrange multiplier) *Let $f \in C^2(\mathbb{R}^n, \mathbb{R})$ a map and $\boldsymbol{x}^\star$ a* local minima *of $f(\boldsymbol{x})$ satisfying the constraints $\boldsymbol{h} \in C^2(\mathbb{R}^n, \mathbb{R}^m)$, i.e. $\boldsymbol{h}(\boldsymbol{x}^\star) = \mathbf{0}$. If $\nabla \boldsymbol{h}(\boldsymbol{x}^\star)$ is* full rank *then there exists m scalars λ_k such that*

$$\nabla_x \mathcal{L}(\boldsymbol{x}^\star, \boldsymbol{\lambda}) = \nabla f(\boldsymbol{x}^\star) - \sum_{k=1}^{m} \lambda_k \nabla h_k(\boldsymbol{x}^\star) = \mathbf{0}^T \tag{A}$$

moreover, for all $\boldsymbol{z} \in \mathbb{R}^n$ which satisfy $\nabla \boldsymbol{h}(\boldsymbol{x}^\star)\boldsymbol{z} = \mathbf{0}$ it follows

$$\boldsymbol{z}^T \nabla_x^2 \mathcal{L}(\boldsymbol{x}^\star, \boldsymbol{\lambda}) \boldsymbol{z} = \boldsymbol{z}^T \left(\nabla^2 f(\boldsymbol{x}^\star) - \sum_{k=1}^{m} \lambda_k \nabla^2 h_k(\boldsymbol{x}^\star) \right) \boldsymbol{z} \geq 0 \tag{B}$$

in other words the matrix $\nabla_x^2 (f(\boldsymbol{x}^\star) - \boldsymbol{\lambda} \cdot \boldsymbol{h}(\boldsymbol{x}^\star))$ is semi-SPD in the Kernel of $\nabla \boldsymbol{h}(\boldsymbol{x}^\star)$.

Proof. Let $\boldsymbol{x}^\star$ a local minima, then there exists $\varepsilon > 0$ such that

$$f(\boldsymbol{x}^\star) \leq f(\boldsymbol{x}), \qquad \text{for all } \boldsymbol{x} \in B \text{ with } \boldsymbol{h}(\boldsymbol{x}) = \boldsymbol{0}, \tag{1}$$

where $B = \{\boldsymbol{x} \mid \left\|\boldsymbol{x} - \boldsymbol{x}^\star\right\| \leq \varepsilon\}$. Consider thus, the functions sequence

$$f_k(\boldsymbol{x}) = f(\boldsymbol{x}) + k\left\|\boldsymbol{h}(\boldsymbol{x})\right\|^2 + \alpha\left\|\boldsymbol{x} - \boldsymbol{x}^\star\right\|^2, \qquad \alpha > 0 \tag{2}$$

with the corresponding sequence of (unconstrained) local minima in B:

$$\boldsymbol{x}_k = \operatorname*{argmin}_{\boldsymbol{x}\in B} f_k(\boldsymbol{x}).$$

The sequence $\boldsymbol{x}_k$ is contained in the compact ball B and from compactness there exists a converging sub-sequence $\boldsymbol{x}_{k_j} \to \bar{\boldsymbol{x}} \in B$. The rest of the proof to verify that $\bar{\boldsymbol{x}} = \boldsymbol{x}^\star$ and it a minimum.

Step 1: $\boldsymbol{h}(\bar{\boldsymbol{x}}) = \boldsymbol{0}$. Notice that the sequence x_k satisfy $f_k(\boldsymbol{x}_k) \leq f(\boldsymbol{x}^\star)$, in fact

$$f_k(\boldsymbol{x}_k) \leq f_k(\boldsymbol{x}^\star) = f(\boldsymbol{x}^\star) + k\left\|\boldsymbol{h}(\boldsymbol{x}^\star)\right\|^2 + \alpha\left\|\boldsymbol{x}^\star - \boldsymbol{x}^\star\right\|^2 = f(\boldsymbol{x}^\star).$$

and by definition (2) we have

$$\begin{aligned} k_j\left\|\boldsymbol{h}(\boldsymbol{x}_{k_j})\right\|^2 + \alpha\left\|\boldsymbol{x}_{k_j} - \boldsymbol{x}^\star\right\|^2 &\leq f(\boldsymbol{x}^\star) - f(\boldsymbol{x}_{k_j}) \\ &\leq f(\boldsymbol{x}^\star) - \min_{\boldsymbol{x}\in B} f(\boldsymbol{x}) = C < +\infty \end{aligned} \tag{3}$$

so that

$$\lim_{j\to\infty}\left\|\boldsymbol{h}(\boldsymbol{x}_{k_j})\right\| = 0 \quad \Rightarrow \quad \left\|\boldsymbol{h}\left(\lim_{j\to\infty}\boldsymbol{x}_{k_j}\right)\right\| = \left\|\boldsymbol{h}\left(\bar{\boldsymbol{x}}\right)\right\| = 0 \quad \Rightarrow \quad \boldsymbol{h}(\bar{\boldsymbol{x}}) = \boldsymbol{0}.$$

Step 2: $\bar{\boldsymbol{x}} = \boldsymbol{x}^\star$. From (3)

$$\alpha\left\|\boldsymbol{x}_{k_j} - \boldsymbol{x}^\star\right\|^2 \leq f(\boldsymbol{x}^\star) - f(\boldsymbol{x}_{k_j}) - k_j\left\|\boldsymbol{h}(\boldsymbol{x}_{k_j})\right\|^2 \leq f(\boldsymbol{x}^\star) - f(\boldsymbol{x}_{k_j})$$

and taking the limit

$$\alpha\left\|\lim_{j\to\infty}\boldsymbol{x}_{k_j} - \boldsymbol{x}^\star\right\|^2 \leq \alpha\left\|\bar{\boldsymbol{x}} - \boldsymbol{x}^\star\right\|^2 \leq f(\boldsymbol{x}^\star) - \lim_{j\to\infty} f(\boldsymbol{x}_{k_j}) \leq f(\boldsymbol{x}^\star) - f(\bar{\boldsymbol{x}})$$

From $\left\|\boldsymbol{h}(\bar{\boldsymbol{x}})\right\| = 0$ it follows that from (1) that $f(\boldsymbol{x}^\star) \leq f(\bar{\boldsymbol{x}})$ and

$$\alpha\left\|\bar{\boldsymbol{x}} - \boldsymbol{x}^\star\right\|^2 \leq f(\boldsymbol{x}^\star) - f(\bar{\boldsymbol{x}}) \leq 0$$

and, thus $\bar{\boldsymbol{x}} = \boldsymbol{x}^\star$.

Step 3: Build multiplier. Cause $\boldsymbol{x}_{k_j}$ are *unconstrained local minima* for $f_{k_j}(\boldsymbol{x})$ it follows

$$\nabla f_{k_j}(\boldsymbol{x}_{k_j}) = \nabla f(\boldsymbol{x}_{k_j}) + k_j \nabla \left\|\boldsymbol{h}(\boldsymbol{x}_{k_j})\right\|^2 + \alpha \nabla \left\|\boldsymbol{x}_{k_j} - \boldsymbol{x}^\star\right\|^2 = \mathbf{0}$$

remembering that

$$\begin{aligned} \nabla \|\boldsymbol{x}\|^2 &= \nabla(\boldsymbol{x} \cdot \boldsymbol{x}) = 2\boldsymbol{x}^T, \\ \nabla \|\boldsymbol{h}(\boldsymbol{x})\|^2 &= \nabla(\boldsymbol{h}(\boldsymbol{x}) \cdot \boldsymbol{h}(\boldsymbol{x})) = 2\boldsymbol{h}(\boldsymbol{x})^T \nabla \boldsymbol{h}(\boldsymbol{x}), \end{aligned}$$

it follows (doing transposition)

$$\nabla f(\boldsymbol{x}_{k_j})^T + 2k_j \nabla \boldsymbol{h}(\boldsymbol{x}_{k_j})^T \boldsymbol{h}(\boldsymbol{x}_{k_j}) + 2\alpha(\boldsymbol{x}_{k_j} - \boldsymbol{x}^\star) = \mathbf{0}. \tag{4}$$

Left multiplying by $\nabla \boldsymbol{h}(\boldsymbol{x}_{k_j})$

$$\nabla \boldsymbol{h}(\boldsymbol{x}_{k_j}) \left[\nabla f(\boldsymbol{x}_{k_j})^T + 2\alpha(\boldsymbol{x}_{k_j} - \boldsymbol{x}^\star)\right] + 2k_j \nabla \boldsymbol{h}(\boldsymbol{x}_{k_j}) \nabla \boldsymbol{h}(\boldsymbol{x}_{k_j})^T \boldsymbol{h}(\boldsymbol{x}_{k_j}) = \mathbf{0}$$

Cause $\nabla \boldsymbol{h}(\boldsymbol{x}^\star) \in \mathbb{R}^{m \times n}$ is full rank for j large by continuity $\nabla \boldsymbol{h}(\boldsymbol{x}_{k_j})$ is full rank and thus $\nabla \boldsymbol{h}(\boldsymbol{x}_{k_j}) \nabla \boldsymbol{h}(\boldsymbol{x}_{k_j})^T \in \mathbb{R}^{m \times m}$ are nonsingular, thus

$$2k_j \boldsymbol{h}(\boldsymbol{x}_{k_j}) = -\left(\nabla \boldsymbol{h}(\boldsymbol{x}_{k_j}) \nabla \boldsymbol{h}(\boldsymbol{x}_{k_j})^T\right)^{-1} \nabla \boldsymbol{h}(\boldsymbol{x}_{k_j}) \left[\nabla f(\boldsymbol{x}_{k_j})^T + 2\alpha(\boldsymbol{x}_{k_j} - \boldsymbol{x}^\star)\right]$$

taking the limit for $j \to \infty$

$$\lim_{j \to \infty} 2k_j \boldsymbol{h}(\boldsymbol{x}_{k_j}) = -\left(\nabla \boldsymbol{h}(\boldsymbol{x}^\star) \nabla \boldsymbol{h}(\boldsymbol{x}^\star)^T\right)^{-1} \nabla \boldsymbol{h}(\boldsymbol{x}^\star) \nabla f(\boldsymbol{x}^\star)^T = -\boldsymbol{\lambda} \tag{5}$$

and taking the limit of (4) with (5) we have $\nabla f(\boldsymbol{x}^\star)^T - \nabla \boldsymbol{h}(\boldsymbol{x}^\star)^T \boldsymbol{\lambda} = \mathbf{0}$.

Step 4: Build a special sequence of $\boldsymbol{z}_j$. We needs a sequence $\boldsymbol{z}_j \to \boldsymbol{z}$ such that $\nabla \boldsymbol{h}(\boldsymbol{x}_{k_j}) \boldsymbol{z}_j = \mathbf{0}$ for all j. The sequence $\boldsymbol{z}_j$ is built as the projection of $\boldsymbol{z}$ into the Kernel of $\nabla \boldsymbol{h}(\boldsymbol{x}_{k_j})$, i.e.

$$\boldsymbol{z}_j = \boldsymbol{z} - \nabla \boldsymbol{h}(\boldsymbol{x}_{k_j})^T \left[\nabla \boldsymbol{h}(\boldsymbol{x}_{k_j}) \nabla \boldsymbol{h}(\boldsymbol{x}_{k_j})^T\right]^{-1} \nabla \boldsymbol{h}(\boldsymbol{x}_{k_j}) \boldsymbol{z}$$

infact

$$\begin{aligned} \nabla \boldsymbol{h}(\boldsymbol{x}_{k_j}) \boldsymbol{z}_j &= \nabla \boldsymbol{h}(\boldsymbol{x}_{k_j}) \boldsymbol{z} - \nabla \boldsymbol{h}(\boldsymbol{x}_{k_j}) \nabla \boldsymbol{h}(\boldsymbol{x}_{k_j})^T \left[\nabla \boldsymbol{h}(\boldsymbol{x}_{k_j}) \nabla \boldsymbol{h}(\boldsymbol{x}_{k_j})^T\right]^{-1} \nabla \boldsymbol{h}(\boldsymbol{x}_{k_j}) \boldsymbol{z} \\ &= \nabla \boldsymbol{h}(\boldsymbol{x}_{k_j}) \boldsymbol{z} - \nabla \boldsymbol{h}(\boldsymbol{x}_{k_j}) \boldsymbol{z} = \mathbf{0} \end{aligned}$$

consider now the limit

$$\begin{aligned} \lim_{j \to \infty} \boldsymbol{z}_j &= \boldsymbol{z} - \lim_{j \to \infty} \nabla \boldsymbol{h}(\boldsymbol{x}_{k_j})^T \left[\nabla \boldsymbol{h}(\boldsymbol{x}_{k_j}) \nabla \boldsymbol{h}(\boldsymbol{x}_{k_j})^T\right]^{-1} \nabla \boldsymbol{h}(\boldsymbol{x}_{k_j}) \boldsymbol{z} \\ &= \boldsymbol{z} - \nabla \boldsymbol{h}(\boldsymbol{x}^\star)^T \left[\nabla \boldsymbol{h}(\boldsymbol{x}^\star) \nabla \boldsymbol{h}(\boldsymbol{x}^\star)^T\right]^{-1} \nabla \boldsymbol{h}(\boldsymbol{x}^\star) \boldsymbol{z} \end{aligned}$$

and thus, if $\boldsymbol{z}$ is in the kernel of $\nabla \boldsymbol{h}(\boldsymbol{x}^\star)$, i.e. $\nabla \boldsymbol{h}(\boldsymbol{x}^\star)\boldsymbol{z} = \boldsymbol{0}$ we have

$$\nabla \boldsymbol{h}(\boldsymbol{x}_{k_j})\boldsymbol{z}_j = \boldsymbol{0} \qquad \text{with} \qquad \lim_{j\to\infty} \boldsymbol{z}_j = \boldsymbol{z}.$$

Step 5: Necessary conditions. Cause $\boldsymbol{x}_{k_j}$ are *unconstrained local minima* for $f_{k_j}(\boldsymbol{x})$ it follows that matrices $\nabla^2 f_{k_j}(\boldsymbol{x}_{k_j})$ are semi positive defined, i.e.

$$\boldsymbol{z}^T \nabla^2 f_{k_j}(\boldsymbol{x}_{k_j})\boldsymbol{z} \geq 0, \qquad \forall \boldsymbol{z} \in \mathbb{R}^n$$

moreover

$$\begin{aligned}
\nabla^2 f_{k_j}(\boldsymbol{x}_{k_j}) &= \nabla^2 f(\boldsymbol{x}_{k_j}) + k\nabla^2 \left\|\boldsymbol{h}(\boldsymbol{x}_{k_j})\right\|^2 + 2\alpha\nabla(\boldsymbol{x}_{k_j} - \boldsymbol{x}^\star) \\
&= \nabla^2 f(\boldsymbol{x}_{k_j})^T + k\nabla^2 \sum_{i=1}^m h_i(\boldsymbol{x}_{k_j})^2 + 2\alpha \boldsymbol{I}
\end{aligned} \tag{6}$$

using the identity

$$\nabla^2 h(\boldsymbol{x})^2 = \nabla(2h(\boldsymbol{x})\nabla h(\boldsymbol{x})^T) = 2\nabla h(\boldsymbol{x})^T \nabla h(\boldsymbol{x}) + 2h(\boldsymbol{x})\nabla^2 h(\boldsymbol{x})$$

in (8)

$$\nabla^2 f_{k_j}(\boldsymbol{x}_{k_j}) = \nabla^2 f(\boldsymbol{x}_{k_j}) + 2k_j \sum_{i=1}^m \nabla h_i(\boldsymbol{x}_{k_j})^T \nabla h_i(\boldsymbol{x}_{k_j}) + 2k_j \sum_{i=1}^m h_i(\boldsymbol{x}_{k_j})\nabla^2 h_i(\boldsymbol{x}_{k_j}) + 2\alpha \boldsymbol{I}$$

Let $\boldsymbol{z} \in \mathbb{R}^n$ then $0 \leq \boldsymbol{z}^T \nabla^2 f_{k_j}(\boldsymbol{x}_{k_j})\boldsymbol{z}$, i.e.

$$0 \leq \boldsymbol{z}^T \nabla^2 f(\boldsymbol{x}_{k_j})\boldsymbol{z} + \sum_{i=1}^m (2k_j h_i(\boldsymbol{x}_{k_j}))\boldsymbol{z}^T \nabla^2 h_i(\boldsymbol{x}_{k_j})\boldsymbol{z} + 2k_j \left\|\nabla \boldsymbol{h}(\boldsymbol{x}_{k_j})\boldsymbol{z}\right\|^2 + 2\alpha \left\|\boldsymbol{z}\right\|^2$$

Inequality is true for all $\boldsymbol{z} \in \mathbb{R}^n$ and thus for any $\boldsymbol{z}$ in the kernel of $\nabla \boldsymbol{h}(\boldsymbol{x}^\star)$. Choosing $\boldsymbol{z}$ in the kernel of $\nabla \boldsymbol{h}(\boldsymbol{x}^\star)$ from previous step the sequence $\boldsymbol{z}_j$ satisfy

$$0 \leq \boldsymbol{z}_j^T \nabla^2 f(\boldsymbol{x}_{k_j})\boldsymbol{z}_j + \sum_{i=1}^m (2k_j h_i(\boldsymbol{x}_{k_j}))\boldsymbol{z}_j^T \nabla^2 h_i(\boldsymbol{x}_{k_j})\boldsymbol{z}_j + 2\alpha \left\|\boldsymbol{z}_j\right\|^2$$

and taking the limit $j \to \infty$ with (5)

$$0 \leq \boldsymbol{z}^T \nabla^2 f(\boldsymbol{x}^\star)\boldsymbol{z} + \sum_{i=1}^m \lambda_i \boldsymbol{z}^T \nabla^2 h_i(\boldsymbol{x}^\star)\boldsymbol{z} + 2\alpha \left\|\boldsymbol{z}\right\|^2$$

cause $\alpha > 0$ can be chosen arbitrarily it follows

$$0 \leq \boldsymbol{z}^T \nabla^2 f(\boldsymbol{x}^\star)\boldsymbol{z} - \sum_{i=1}^{m} \lambda_i \left[\boldsymbol{z}^T \nabla^2 h_i(\boldsymbol{x}^\star)\boldsymbol{z}\right]$$

which is the relation to be proved. □

Inequality constraints

It is possible to adapt theorem 1 for inequality constraints. Consider the NLP problem

$$\begin{aligned} &\text{minimize:} && f(\boldsymbol{x}) \\ &\text{subject to:} && h_i(\boldsymbol{x}) = 0 && i = 1, 2, \ldots, m \\ & && g_i(\boldsymbol{x}) \geq 0 && i = 1, 2, \ldots, p \end{aligned}$$

introducing the *slack* variables e_i, $i = 1, 2, \ldots, p$ and $\boldsymbol{y}^T = (\boldsymbol{x}^T, \boldsymbol{e}^T)$ the new problem

$$\begin{aligned} &\text{minimize:} && \mathsf{f}(\boldsymbol{y}) = f(\boldsymbol{x}) \\ &\text{subject to:} && \mathsf{h}_i(\boldsymbol{y}) = h_i(\boldsymbol{x}) = 0 && i = 1, 2, \ldots, m \\ & && \mathsf{h}_{i+m}(\boldsymbol{y}) = g_i(\boldsymbol{x}) - e_i^2 = 0 && i = 1, 2, \ldots, p \end{aligned}$$

with the Lagrangian function:

$$\mathcal{L}(\boldsymbol{x}, \boldsymbol{e}, \boldsymbol{\lambda}, \boldsymbol{\mu}) = f(\boldsymbol{x}) - \sum_{k=1}^{m} \lambda_k h_k(\boldsymbol{x}) - \sum_{k=1}^{p} \mu_k \left(g_k(\boldsymbol{x}) - e_k^2\right)$$

The first order condition becomes

$$\begin{aligned} \nabla_x \mathcal{L}(\boldsymbol{x}^\star, \boldsymbol{e}, \boldsymbol{\lambda}, \boldsymbol{\mu}) &= \nabla f(\boldsymbol{x}^\star) - \sum_{k=1}^{m} \lambda_k \nabla h_k(\boldsymbol{x}^\star) - \sum_{k=1}^{p} \mu_k \nabla g_k(\boldsymbol{x}^\star) = \boldsymbol{0}^T, \\ \nabla_e \mathcal{L}(\boldsymbol{x}^\star, \boldsymbol{e}, \boldsymbol{\lambda}, \boldsymbol{\mu}) &= 2(\mu_1 e_1, \ldots, \mu_p e_p) = \boldsymbol{0}^T, \\ h_k(\boldsymbol{x}^\star) &= 0, \\ g_k(\boldsymbol{x}^\star) &= e_k^2 \geq 0, \end{aligned}$$

and second order condition become $\boldsymbol{z}^T \nabla^2_{(x,e)} \mathcal{L}(\boldsymbol{x}^\star, \boldsymbol{e}, \boldsymbol{\lambda}, \boldsymbol{\mu})\boldsymbol{z} \geq 0$ for $\boldsymbol{z}$ in the kernel of matrix

$$\begin{pmatrix} \nabla_x \boldsymbol{h}(\boldsymbol{x}^\star) & \boldsymbol{0} \\ \nabla_x \boldsymbol{g}(\boldsymbol{x}^\star) & 2\,\mathrm{diag}(e_1, \ldots, e_p) \end{pmatrix} \tag{7}$$

where

$$\nabla^2_{(x,e)}\mathcal{L}(\boldsymbol{x}^\star,\boldsymbol{e},\boldsymbol{\lambda},\boldsymbol{\mu})\boldsymbol{z} = \begin{pmatrix} \nabla^2_x\mathcal{L}(\boldsymbol{x}^\star,\boldsymbol{e},\boldsymbol{\lambda},\boldsymbol{\mu}) & \mathbf{0} \\ \mathbf{0} & \nabla^2_e\mathcal{L}(\boldsymbol{x}^\star,\boldsymbol{e},\boldsymbol{\lambda},\boldsymbol{\mu}) \end{pmatrix} \tag{8}$$

and $\nabla_x\nabla_e^T\mathcal{L}(\boldsymbol{x}^\star,\boldsymbol{e},\boldsymbol{\lambda},\boldsymbol{\mu}) = \mathbf{0}$, moreover

$$\begin{aligned}
\nabla^2_x\mathcal{L}(\boldsymbol{x}^\star,\boldsymbol{e},\boldsymbol{\lambda},\boldsymbol{\mu}) &= \nabla^2 f(\boldsymbol{x}^\star) - \sum_{k=1}^{m}\lambda_k\nabla^2 h_k(\boldsymbol{x}^\star) - \sum_{k=1}^{p}\mu_k\nabla^2 g_k(\boldsymbol{x}^\star), \\
\nabla^2_e\mathcal{L}(\boldsymbol{x}^\star,\boldsymbol{e},\boldsymbol{\lambda},\boldsymbol{\mu}) &= 2\,\mathrm{diag}(\mu_1,\mu_2,\ldots,\mu_p).
\end{aligned}$$

Notice that $\mu_k e_k = 0$ is equivalent of $\mu_k e_k^2 = 0$ and thus $\mu_k g_k(\boldsymbol{x}^\star) = 0$. So that when $g_k(\boldsymbol{x}^\star) > 0$ then $\mu_k = 0$. Up to a reordering we split $\boldsymbol{g}(\boldsymbol{x}) = \begin{pmatrix}\boldsymbol{g}^{(1)}(\boldsymbol{x}) \\ \boldsymbol{g}^{(2)}(\boldsymbol{x})\end{pmatrix}$ where

$$\begin{aligned}
g_k(\boldsymbol{x}^\star) &= e_k^2 = 0, \qquad k = 1,2,\ldots,r \\
g_k(\boldsymbol{x}^\star) &= e_k^2 > 0, \qquad k = r+1, r+2,\ldots,p
\end{aligned}$$

and thus (7) becomes

$$\begin{pmatrix} \nabla_x\boldsymbol{h}(\boldsymbol{x}^\star) & \mathbf{0} & \mathbf{0} \\ \nabla_x\boldsymbol{g}^{(1)}(\boldsymbol{x}^\star) & \mathbf{0} & \mathbf{0} \\ \nabla_x\boldsymbol{g}^{(2)}(\boldsymbol{x}^\star) & \mathbf{0} & \boldsymbol{E} \end{pmatrix}, \qquad 2\,\mathrm{diag}(e_{k+1},\ldots,e_p) = \boldsymbol{E}. \tag{9}$$

and

$$\nabla^2_e\mathcal{L}(\boldsymbol{x}^\star,\boldsymbol{e},\boldsymbol{\lambda},\boldsymbol{\mu}) = \begin{pmatrix} \boldsymbol{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, \qquad \boldsymbol{M} = 2\,\mathrm{diag}(\mu_1,\mu_2,\ldots,\mu_r) \tag{10}$$

The group of constraints $\boldsymbol{g}^{(1)}(\boldsymbol{x}^\star)$ that are zeros are the active constraints. The kernel of (9) can be written as

$$\mathcal{K} = \begin{pmatrix} \boldsymbol{K} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{I} \\ -\boldsymbol{E}^{-1}\nabla_x\boldsymbol{g}^{(2)}(\boldsymbol{x}^\star)\boldsymbol{K} & \mathbf{0} \end{pmatrix}, \qquad \boldsymbol{K} \text{ is the kernel of } \begin{pmatrix} \nabla_x\boldsymbol{h}(\boldsymbol{x}^\star) \\ \nabla_x\boldsymbol{g}^{(1)}(\boldsymbol{x}^\star) \end{pmatrix} \tag{11}$$

where $\boldsymbol{K}$ is the kernel of the matrix

$$\begin{pmatrix} \nabla_x\boldsymbol{h}(\boldsymbol{x}^\star) \\ \nabla_x\boldsymbol{g}^{(1)}(\boldsymbol{x}^\star) \end{pmatrix}$$

thus $\boldsymbol{z}$ can be written as $\boldsymbol{\mathcal{K}}\boldsymbol{d}$ and thus second order condition $\boldsymbol{z}^T\nabla^2_{(x,e)}\mathcal{L}(\boldsymbol{x}^\star, \boldsymbol{e}, \boldsymbol{\lambda}, \boldsymbol{\mu})\boldsymbol{z} \geq 0$ become

$$0 \leq \boldsymbol{d}^T \left[\boldsymbol{\mathcal{K}}^T \nabla^2_{(x,e)}\mathcal{L}(\boldsymbol{x}^\star, \boldsymbol{e}, \boldsymbol{\lambda}, \boldsymbol{\mu})\boldsymbol{\mathcal{K}}\right] \boldsymbol{d}, \qquad \boldsymbol{d} \in \mathbb{R}^s$$

and using (11) with (8) and (10)

$$\begin{aligned}\left[\boldsymbol{\mathcal{K}}^T \nabla^2_{(x,e)}\mathcal{L}(\boldsymbol{x}^\star, \boldsymbol{e}, \boldsymbol{\lambda}, \boldsymbol{\mu})\boldsymbol{\mathcal{K}}\right] &= \boldsymbol{\mathcal{K}}^T \begin{pmatrix} \nabla^2_x \mathcal{L}(\boldsymbol{x}^\star, \boldsymbol{e}, \boldsymbol{\lambda}, \boldsymbol{\mu}) & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{M} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \end{pmatrix} \boldsymbol{\mathcal{K}}, \\ &= \begin{pmatrix} \boldsymbol{K}^T \nabla^2_x \mathcal{L}(\boldsymbol{x}^\star, \boldsymbol{e}, \boldsymbol{\lambda}, \boldsymbol{\mu})\boldsymbol{K} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{M} \end{pmatrix}\end{aligned}$$

Using the solution algorithm of the equality constrained problem we have

- Necessary condition: the matrices

  $$\boldsymbol{K}^T \nabla^2_x \mathcal{L}(\boldsymbol{x}^\star, \boldsymbol{e}, \boldsymbol{\lambda}, \boldsymbol{\mu})\boldsymbol{K}, \qquad \text{and} \qquad \boldsymbol{M}$$

  must be semi-positive defined. This imply that $\mu_k \geq 0$ for $k = 1, 2, \ldots, p$

- Sufficient condition: the matrices

  $$\boldsymbol{K}^T \nabla^2_x \mathcal{L}(\boldsymbol{x}^\star, \boldsymbol{e}, \boldsymbol{\lambda}, \boldsymbol{\mu})\boldsymbol{K}, \qquad \text{and} \qquad \boldsymbol{M}$$

  must be positive defined. This imply that $\mu_k > 0$ for the active constraints.

Constrained minima, NLP problem

Consider the constrained minimization problem

$$\begin{aligned} \text{minimize:} \quad & f(\boldsymbol{x}) & \\ \text{subject to:} \quad & h_i(\boldsymbol{x}) = 0 & i = 1,2,\ldots,m \\ & g_i(\boldsymbol{x}) \geq 0 & i = 1,2,\ldots,p \end{aligned} \tag{12}$$

Solution algorithm

- Compute the Lagrangian function:

$$\mathcal{L}(\boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}) = f(\boldsymbol{x}) - \sum_{k=1}^{m} \lambda_k h_k(\boldsymbol{x}) - \sum_{k=1}^{p} \mu_k g_k(\boldsymbol{x})$$

- Solve the nonlinear system

$$\begin{aligned} \nabla_x \mathcal{L}(\boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}) &= \mathbf{0}^T \\ h_k(\boldsymbol{x}) &= 0 \qquad k = 1,2,\ldots,m \\ \mu_k g_k(\boldsymbol{x}) &= 0 \qquad k = 1,2,\ldots,p \end{aligned}$$

  keep only the solutions with $\mu_k^\star \geq 0$ and $g_k(\boldsymbol{x}^\star) \geq 0$.

- For each solution points $(\boldsymbol{x}^\star, \boldsymbol{\lambda}^\star, \boldsymbol{\mu}^\star)$ compute $\nabla \boldsymbol{h}(\boldsymbol{x}^\star)$ with $\nabla g_k(\boldsymbol{x}^\star)$ where $g_k(\boldsymbol{x}^\star) = 0$ are the active constraints with $\mu_k > 0$ and check they are linearly independent.

- Compute matrix $\boldsymbol{K}$ the kernel of $\nabla \boldsymbol{h}(\boldsymbol{x}^\star)$ with $\nabla g_k(\boldsymbol{x}^\star)$ where $g_k(\boldsymbol{x}^\star) = 0$ are the active constraints with $\mu_k > 0$.

- Compute the reduce Hessian

$$\boldsymbol{H} = \boldsymbol{K}^T \nabla_x^2 \mathcal{L}(\boldsymbol{x}^\star, \boldsymbol{\lambda}^\star) \boldsymbol{K},$$

  - Necessary condition: $\boldsymbol{H}$ is semi-positive definite.
  - Sufficient condition: $\boldsymbol{H}$ is positive definite and $\mu_k > 0$ for all the active constraints.

Definition 1 *The set*

$$\mathcal{F} = \{\boldsymbol{x} \in \mathbb{R}^n \mid h_k(\boldsymbol{x}) = 0, \quad k = 1,2,\ldots,m, \qquad g_k(\boldsymbol{x}) \geq 0, \quad k = 1,2,\ldots,p,\}$$

is called the feasible region *or set of* feasible points.

Definition 2 (Active set) *The set* $\mathcal{A}(\boldsymbol{x})$ *defined as*

$$\mathcal{A}(\boldsymbol{x}) = \{k \mid g_k(\boldsymbol{x}) = 0\}$$

is the set of active (unilateral) constraints.

Constrained minima general theorem and KKT

The following theorem (see [1]) give the necessary conditions for constrained minima. Notice that no condition on the constraints are necessary.

Theorem 2 (Fritz John) *If the functions* $f(\boldsymbol{x})$, $g_1(\boldsymbol{x})$,. . . , $g_p(\boldsymbol{x})$, *are differentiable, then a necessary condition that* $\boldsymbol{x}^\star$ *be a local minimum to problem:*

$$\begin{aligned} &\textit{minimize:} && f(\boldsymbol{x}) \\ &\textit{subject to:} && g_i(\boldsymbol{x}) \geq 0 \qquad i = 1, 2, \ldots, p \end{aligned}$$

is that there exist scalars $\mu_0^\star$, $\mu_1^\star$, $\mu_p^\star$, *(not all zero) such that the following inequalities and equalities are satisfied:*

$$\begin{aligned} \nabla_x \mathcal{L}(\boldsymbol{x}^\star, \boldsymbol{\mu}^*) &= \mathbf{0}^T \\ \mu_k^* g_k(\boldsymbol{x}^\star) &= 0, \qquad k = 1, 2, \ldots, p; \\ \mu_k^* &\geq 0, \qquad k = 0, 1, 2, \ldots, p; \end{aligned}$$

where

$$\mathcal{L}(\boldsymbol{x}, \boldsymbol{\mu}) = f(\boldsymbol{x}) - \sum_{k=1}^{p} \mu_k \, g_k(\boldsymbol{x})$$

In [2] Kuhn and Tucker showed that if a condition, called the *first order constraint qualification*, holds at $\boldsymbol{x}^\star$, $\boldsymbol{\lambda}^\star$ then λ_0 can be taken equal to 1.

Definition 3 (Constraints qualification LI) *Let be the unilateral and bilateral constraints* $\boldsymbol{g}(\boldsymbol{x})$ *and* $\boldsymbol{h}(\boldsymbol{x})$, *the point* $\boldsymbol{x}^\star$ *is admissible if*

$$g_k(\boldsymbol{x}^\star) \geq 0, \qquad h_k(\boldsymbol{x}^\star) = 0.$$

The constraints $\boldsymbol{g}(\boldsymbol{x})$ *and* $\boldsymbol{h}(\boldsymbol{x})$ *are* qualified *at* $\boldsymbol{x}^\star$ *if the point* $\boldsymbol{x}^\star$ *is admissible and the vectors*

$$\{\nabla g_k(\boldsymbol{x}^\star) \ : \ k \in \mathcal{A}(\boldsymbol{x}^\star)\} \bigcup \{\nabla h_1(\boldsymbol{x}^\star), \nabla h_2(\boldsymbol{x}^\star), \ldots, \nabla h_m(\boldsymbol{x}^\star)\}$$

are linearly independent.

Definition 4 (Constraint qualification (Mangasarian-Fromovitz)) *The constraints* $\boldsymbol{g}(\boldsymbol{x})$ *and* $\boldsymbol{h}(\boldsymbol{x})$ *are* qualified *at* $\boldsymbol{x}^\star$ *if the point* $\boldsymbol{x}^\star$ *is admissible and* ***does not exists*** *a linear combination*

$$\sum_{k\in\mathcal{A}(\boldsymbol{x}^\star)}^{m} \alpha_k \nabla g_k(\boldsymbol{x}^\star) + \sum_{k=1}^{m} \beta_k \nabla h_k(\boldsymbol{x}^\star) = \mathbf{0}$$

with $\alpha_k \geq 0$ *for* $k \in \mathcal{A}(\boldsymbol{x}^\star)$ *and* α_k *with* β_k *not all* 0*. In other words, there not exists a non trivial linear combination of the null vector such that* $\alpha_k \geq 0$ *for* $k \in \mathcal{A}(\boldsymbol{x}^\star)$*.*

The next theorems are taken from [3].

Theorem 3 (First order necessary conditions) *Let* $f \in C^1(\mathbb{R}^n)$ *and the constraints* $\boldsymbol{g} \in C^1(\mathbb{R}^n, \mathbb{R}^p)$ *and* $\boldsymbol{h} \in C^1(\mathbb{R}^n, \mathbb{R}^m)$*. Suppose that* $\boldsymbol{x}^\star$ *is a local minima of* (12) *and that the constraints qualification LI holds at* $\boldsymbol{x}^\star$*. Then there are Lagrange multiplier vectors* $\boldsymbol{\lambda}$ *and* $\boldsymbol{\mu}$ *such that the following conditions are satisfied at* $(\boldsymbol{x}^\star, \boldsymbol{\lambda}, \boldsymbol{\mu})$

$$\begin{aligned} \nabla_x \mathcal{L}(\boldsymbol{x}^\star, \boldsymbol{\lambda}^*, \boldsymbol{\mu}^*) &= \mathbf{0}^T \\ h_k(\boldsymbol{x}^\star) &= 0, \qquad k = 1, 2, \ldots, m; \\ \mu_k^* g_k(\boldsymbol{x}^\star) &= 0, \qquad k = 1, 2, \ldots, p; \\ \mu_k^* &\geq 0, \qquad k = 1, 2, \ldots, p; \end{aligned}$$

where

$$\mathcal{L}(\boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}) = f(\boldsymbol{x}) - \sum_{k=1}^{m} \lambda_k h_k(\boldsymbol{x}) - \sum_{k=1}^{p} \mu_k g_k(\boldsymbol{x})$$

Theorem 4 (Second order necessary conditions) *Let* $f \in C^2(\mathbb{R}^n)$ *and the constraints* $\boldsymbol{g} \in C^2(\mathbb{R}^n, \mathbb{R}^p)$ *and* $\boldsymbol{h} \in C^2(\mathbb{R}^n, \mathbb{R}^m)$*. Let* $\boldsymbol{x}^\star$ *satisfying the First order necessary conditions, a* ***necessary*** *condition for* $\boldsymbol{x}^\star$ *be a* local minima *is that the* $m + p$ *scalars (Lagrange Multiplier) of the first order necessary condition satisfy:*

$$\boldsymbol{d}^T \nabla_x^2 \mathcal{L}(\boldsymbol{x}^\star, \boldsymbol{\lambda}^*, \boldsymbol{\mu}^*) \boldsymbol{d} \geq 0$$

for all $\boldsymbol{d}$ *such that*

$$\begin{aligned} &\nabla h_k(\boldsymbol{x}^\star)\boldsymbol{d} = 0, && k = 1, 2, \ldots, m \\ &\nabla g_k(\boldsymbol{x}^\star)\boldsymbol{d} = 0, && \text{if } k \in \mathcal{A}(\boldsymbol{x}^\star) \text{ and } \mu_k > 0 \\ &\nabla g_k(\boldsymbol{x}^\star)\boldsymbol{d} \geq 0, && \text{if } k \in \mathcal{A}(\boldsymbol{x}^\star) \text{ and } \mu_k = 0 \end{aligned}$$

Remark 1 *The conditions*

$$\nabla g_k(\boldsymbol{x}^\star)\boldsymbol{d} = 0, \qquad \textit{if } k \in \mathcal{A}(\boldsymbol{x}^\star) \textit{ and } \mu_k > 0$$
$$\nabla g_k(\boldsymbol{x}^\star)\boldsymbol{d} \geq 0, \qquad \textit{if } k \in \mathcal{A}(\boldsymbol{x}^\star) \textit{ and } \mu_k = 0$$

restrict the space of direction to be considered. If changed with

$$\nabla g_k(\boldsymbol{x}^\star)\boldsymbol{d} = 0, \qquad \textit{if } k \in \mathcal{A}(\boldsymbol{x}^\star)$$

theorems 4 is still valid cause necessary condition is tested in a smaller set.

Theorem 5 (Second order sufficient conditions) *Let $f \in C^2(\mathbb{R}^n)$ and the constraints $\boldsymbol{g} \in C^2(\mathbb{R}^n, \mathbb{R}^p)$ and $\boldsymbol{h} \in C^2(\mathbb{R}^n, \mathbb{R}^m)$. Let $\boldsymbol{x}^\star$ satisfying the First order necessary conditions, a **sufficient** condition for $\boldsymbol{x}^\star$ be a* local minima *is that the $m+p$ scalars (Lagrange Multiplier) of the first order necessary condition satisfy:*

$$\boldsymbol{d}^T \nabla_x^2 \mathcal{L}(\boldsymbol{x}^\star, \boldsymbol{\lambda}^*, \boldsymbol{\mu}^*)\boldsymbol{d} > 0$$

for all $\boldsymbol{d} \neq \boldsymbol{0}$ such that

$$\nabla h_k(\boldsymbol{x}^\star)\boldsymbol{d} = 0, \qquad k = 1, 2, \ldots, m$$
$$\nabla g_k(\boldsymbol{x}^\star)\boldsymbol{d} = 0, \qquad \textit{if } k \in \mathcal{A}(\boldsymbol{x}^\star) \textit{ and } \mu_k > 0$$
$$\nabla g_k(\boldsymbol{x}^\star)\boldsymbol{d} \geq 0, \qquad \textit{if } k \in \mathcal{A}(\boldsymbol{x}^\star) \textit{ and } \mu_k = 0$$

Remark 2 *The condition*

$$\nabla g_k(\boldsymbol{x}^\star)\boldsymbol{d} \geq 0, \qquad \textit{if } k \in \mathcal{A}(\boldsymbol{x}^\star) \textit{ and } \mu_k = 0$$

restrict the space of direction to be considered. If omitted the theorems 5 is still valid cause sufficient condition is tested in a larger set.

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