Numerical Methods for Dynamic System and Control del 14/1/2014

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1

Study the following constrained minimization problem.

$$\text{minimize} \quad f(x,y,z) = z^2 - x - y, \qquad \text{subject to} \qquad x + 2\,z = 1, \qquad x \geq 0, \qquad y \leq 1,$$

KKT system of first order condition:

$$\begin{cases} -\mu_2 - \lambda - 1 &= 0 \\ -\mu_1 - 1 &= 0 \\ -2\lambda + 2z &= 0 \\ x + 2z - 1 &= 0 \\ \mu_1(1-y) &= 0 \\ \mu_2 x &= 0 \end{cases}$$

Solutions of KKT system:

$$\begin{cases} x = 0, \quad y = 1, \quad z = 1/2, \quad \lambda = 1/2, \quad \mu_1 = 1, \quad \mu_2 = -3/2 & \text{NO } (\mu_2 < 0) \\ x = 3, \quad y = 1, \quad z = -1, \quad \lambda = -1, \quad \mu_1 = 1, \quad \mu_2 = 0 & \text{SI} \end{cases}$$

Discussion of the stationary point: $x = 3, \quad y = 1, \quad z = -1, \quad \lambda = -1, \quad \mu_1 = 1, \quad \mu_2 = 0$

$$\nabla[h, g_1] = \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 0 \end{pmatrix} \qquad \boldsymbol{K} = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \qquad \boldsymbol{K}^T \nabla^2 \mathcal{L} \boldsymbol{K} = [2]$$

2

Solve the following recurrence

$$x_{k+1} = x_k - k, \qquad x_0 = 1,$$
$$y_{k+1} = x_k + y_k, \qquad y_0 = 1,$$

$\mathcal{Z}-$transform:

$$(\zeta - 1)x(\zeta) - \zeta = \frac{\zeta}{(\zeta-1)^2},$$

$$(\zeta - 1)y(\zeta) - \zeta = x(\zeta),$$

Solution in $\mathcal{Z}$:

$$x(\zeta) = \zeta^2 \frac{\zeta - 2}{(\zeta-1)^3},$$

$$y(\zeta) = \zeta \frac{\zeta^3 - 2\zeta^2 + \zeta - 1}{(\zeta-1)^4},$$

Solution in k

$$x_k = 1 - \frac{1}{2}k(k-1),$$

$$y_k = 1 + \frac{2}{3}k + \frac{1}{2}k^2 - \frac{1}{6}k^3,$$

3

Solve using Laplace transform the following boundary value problem

$$x''(t) + y'(t) = -1$$
$$y''(t) - x'(t) = -1$$
$$x(0) = 0, \quad y(0) = 0, \quad x(1) = 1, \quad y(1) = -1,$$

Suggestion: Set $A = x'(0)$ and $B = y'(0)$ then using Laplace transform compute A and B which satisfy the boundary conditions.

The Laplace transform:

$$\begin{cases} s^2 x(s) - A + s y(s) &= -\dfrac{1}{s} \\ s^2 y(s) - B - s x(s) &= -\dfrac{1}{s} \end{cases}$$

Solution in s with partial fraction expansion:

$$\begin{cases} x(s) &= \dfrac{As^2 - Bs - s + 1}{s^2(s^2+1)} &= \dfrac{1}{s^2} - \dfrac{1+B}{s} + \dfrac{Bs + A + s - 1}{s^2+1} \\ y(s) &= \dfrac{Bs^2 + As - s - 1}{s^2(s^2+1)} &= \dfrac{A-1}{s} - \dfrac{1}{s^2} + \dfrac{-As + B + s + 1}{s^2+1} \end{cases}$$

Value of A and B:

$A = 1, \qquad B = -1$

Solution in t:

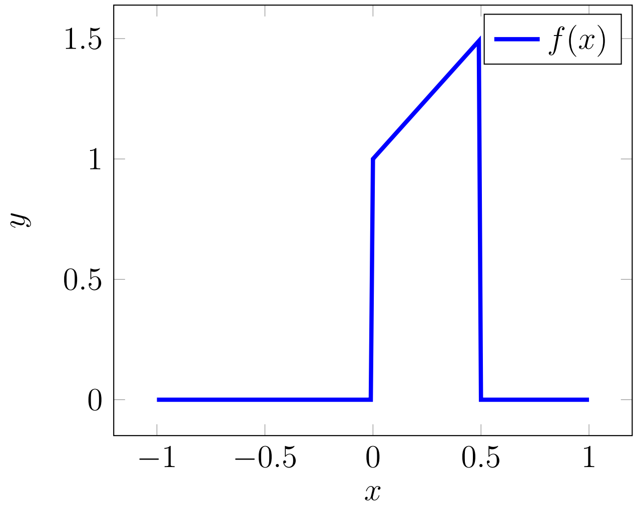
$$\begin{cases} x(t) &= t \\ y(t) &= -t \end{cases}$$

4

Compute the coefficients of the Fourier series of the following function:

$$f(x) = \begin{cases} 0 & \text{for } -1 \le x < 0 \\ 1 + x & \text{for } 0 \le x < 1/2 \\ 0 & \text{for } 1/2 \le x < 1 \end{cases}$$

defined for $x \in (-1, 1)$ and extended periodically.



Coefficients a_k and b_k for the Fourier serie:

$$a_0 = \frac{5}{8}$$

$$a_k = \frac{3\sin(k\pi/2)}{2k\pi} + \frac{\cos(k\pi/2)}{k^2\pi^2} - \frac{1}{k^2\pi^2} = \frac{1}{2k^2\pi^2} \begin{cases} 0 & \text{for } k = 4m \\ 3\pi k - 2 & \text{for } k = 4m+1 \\ -2 & \text{for } k = 4m+2 \\ -3\pi k - 2 & \text{for } k = 4m+3 \end{cases}$$

$$b_k = -\frac{3\cos(k\pi/2)}{2k\pi} + \frac{\sin(k\pi/2)}{k^2\pi^2} + \frac{1}{k\pi}$$