Numerical Methods for Dynamic System and Control del 7/7/2014

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1

Solve the following optimal control problem

$$
\begin{cases}
\text{Minimize:} \quad \displaystyle\int_0^1 u(t)^2 \,\mathrm{d}t \\
\displaystyle\int_0^1 x(t)\,\mathrm{d}t = 0 \\
x'(t) = x(t) + u(t) \\
x(0) = 0,
\end{cases}
$$

The Boundary Value Problem (BVP):
The optimal control $u(t)$:
The solution of the BVP:

2

Study the following constrained minimization problem.

$$\text{minimize} \quad f(x,y,z) = z + y^2 - x^2, \qquad \text{subject to} \qquad x = z, \qquad x \geq 0, \qquad x \leq 1,$$

KKT system of first order condition:

$$\begin{cases} -\mu_1 + \mu_2 - \lambda - 2x &= 0 \\ 2y &= 0 \\ \lambda + 1 &= 0 \\ x - z &= 0 \\ x\mu_1 &= 0 \\ (1-x)\mu_2 &= 0 \end{cases}$$

Solutions of KKT system:

$$\begin{cases} x = 0, \quad y = 0, \quad z = 0, \quad \lambda = -1, \quad \mu_1 = 1, \quad \mu_2 = 0 \\ x = 1, \quad y = 0, \quad z = 1, \quad \lambda = -1, \quad \mu_1 = 0, \quad \mu_2 = 1 \\ x = 1/2, \quad y = 0, \quad z = 1/2, \quad \lambda = -1, \quad \mu_1 = 0, \quad \mu_2 = 0 \end{cases}$$

Discussion of the stationary point: $x = 0, \quad y = 0, \quad z = 0, \quad \lambda = -1, \quad \mu_1 = 1, \quad \mu_2 = 0$

$$\nabla[h, g_1] = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix} \qquad \boldsymbol{K} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad \boldsymbol{K}^T \nabla^2 \mathcal{L} \boldsymbol{K} = [2]$$

3

Solve the following recurrence

$$x_{k+1} = y_k + 1, \qquad x_0 = 0,$$
$$y_{k+1} = 2x_k - 1, \qquad y_0 = 1,$$

$\mathcal{Z}-$transform:

$$\begin{aligned} \zeta x(\zeta) &= y(\zeta) + \frac{\zeta}{\zeta - 1}, \\ \zeta y(\zeta) - \zeta &= 2x(\zeta) - \frac{\zeta}{\zeta - 1}, \end{aligned}$$

Solution in $\mathcal{Z}$:

$$\begin{aligned} x(\zeta) &= \frac{2\zeta}{\zeta^2 - 2}, \\ y(\zeta) &= \zeta \frac{\zeta^2 - 2\zeta + 2}{(\zeta^2 - 2)(\zeta - 1)} = \frac{2\zeta^2}{\zeta^2 - 2} - \frac{\zeta}{\zeta - 1} \end{aligned}$$

Solution in k

$$\begin{aligned} x_k &= \frac{1}{2}(-\sqrt{2})^{k+1} + \frac{1}{2}(\sqrt{2})^{k+1}, \\ y_k &= (-\sqrt{2})^k + (\sqrt{2})^k - 1, \end{aligned}$$

4

Solve using Laplace transform the following boundary value problem

$$\begin{aligned} x''(t) + y(t) &= 1 - \cos t \\ y''(t) - x''(t) &= \cos t \end{aligned}$$

$$x(0) = 1, \quad y(0) = 1, \quad x(\pi/2) = 0, \quad y(\pi/2) = 1,$$

Suggestion: Set $A = x'(0)$ and $B = y'(0)$ then using Laplace transform compute A and B which satisfy the boundary conditions.

The Laplace transform:

$$\begin{cases} s^2 x(s) - A - s + y(s) &= \dfrac{1}{s} - \dfrac{s}{s^2+1} \\ s^2 y(s) - B - s^2 x(s) + A &= \dfrac{s}{s^2+1} \end{cases}$$

Solution in s with partial fraction expansion:

$$\begin{cases} x(s) &= \dfrac{s^3 + As^2 + A - B}{s^2(s^2+1)} &= \dfrac{B+s}{s^2+1} + \dfrac{A-B}{s^2} \\ y(s) &= \dfrac{s^2 + Bs + 1}{s^2(s^2+1)} &= \dfrac{B}{s^2+1} + \dfrac{1}{s} \end{cases}$$

Value of A and B:

$A = 0, \qquad B = 0$

Solution in t:

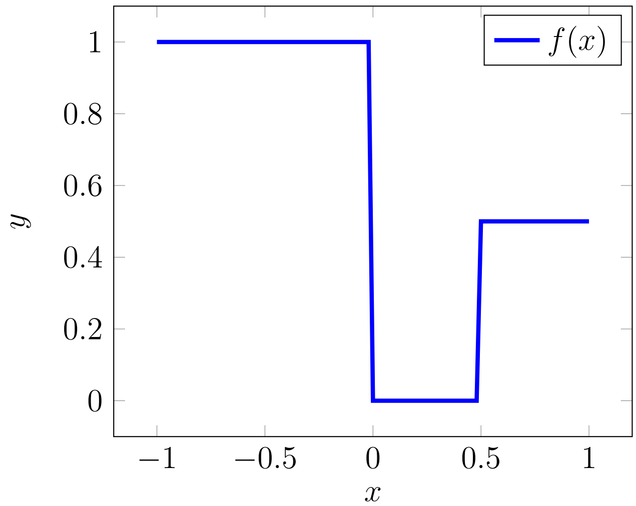
$$\begin{cases} x(t) &= \cos t \\ y(t) &= 1 \end{cases}$$

5

Compute the coefficients of the Fourier series of the following function:

$$f(x) = \begin{cases} 1 & \text{for } -1 \leq x < 0 \\ 0 & \text{for } 0 \leq x < 1/2 \\ 1/2 & \text{for } 1/2 \leq x < 1 \end{cases}$$

defined for $x \in (-1, 1)$ and extended periodically.



Coefficients a_k and b_k for the Fourier serie:

$$a_0 = \frac{5}{4}$$

$$a_k = -\frac{\sin(k\pi/2)}{2k\pi} = \frac{1}{2k\pi} \begin{cases} 0 & \text{for } k = 4m \\ -1 & \text{for } k = 4m+1 \\ 0 & \text{for } k = 4m+2 \\ 1 & \text{for } k = 4m+3 \end{cases}$$

$$b_k = -\frac{\cos(k\pi) + \cos(k\pi/2) - 2}{2k\pi} = \frac{1}{2k\pi} \begin{cases} 0 & \text{for } k = 4m \\ -3 & \text{for } k = 4m+1 \\ -2 & \text{for } k = 4m+2 \\ -3 & \text{for } k = 4m+3 \end{cases}$$