# Metodo di eliminazione di Gauss

$$\begin{cases} y + 2z + w = 2 \\ x + 2y + 2z - w = 5 \\ 3y + 2z + 2w = 6 \\ x + 4y + 2z + 3w = 9 \end{cases} \longrightarrow \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \left[\begin{array}{cccc|c} 0 & 1 & 2 & 1 & 2 \\ 1 & 2 & 2 & -1 & 5 \\ 0 & 3 & 2 & 2 & 6 \\ 1 & 4 & 2 & 3 & 9 \end{array}\right]$$

Scambio riga 1 con riga 2

$$\begin{array}{c} 2 \\ 1 \\ 3 \\ 4 \end{array} \left[\begin{array}{cccc|c} 1 & 2 & 2 & -1 & 5 \\ 0 & 1 & 2 & 1 & 2 \\ 0 & 3 & 2 & 2 & 6 \\ 1 & 4 & 2 & 3 & 9 \end{array}\right] \begin{array}{l} \\ \text{ok} \\ \text{ok} \\ (4) \leftarrow (4) - (1) \end{array} \longrightarrow \begin{array}{c} 2 \\ 1 \\ 3 \\ 4 \end{array} \left[\begin{array}{cccc|c} 1 & 2 & 2 & -1 & 5 \\ 0 & 1 & 2 & 1 & 2 \\ 0 & 3 & 2 & 2 & 6 \\ 1 & 2 & 0 & 4 & 4 \end{array}\right]$$

$$\begin{array}{c} 2 \\ 1 \\ 3 \\ 4 \end{array} \left[\begin{array}{cccc|c} 1 & 2 & 2 & -1 & 5 \\ 0 & 1 & 2 & 1 & 2 \\ 0 & 3 & 2 & 2 & 6 \\ 1 & 2 & 0 & 4 & 4 \end{array}\right] \qquad \begin{array}{l} (3) \leftarrow (3) - 3(2) \\ (4) \leftarrow (4) - 2(2) \end{array} \qquad \begin{array}{c} 2 \\ 1 \\ 3 \\ 4 \end{array} \left[\begin{array}{cccc|c} 1 & 2 & 2 & -1 & 5 \\ 0 & 1 & 2 & 1 & 2 \\ 0 & 3 & -4 & -1 & 0 \\ 1 & 2 & -4 & 2 & 0 \end{array}\right]$$

$$(4) \leftarrow (4) - (3)$$

$$\begin{array}{c} 2 \\ 1 \\ 3 \\ 4 \end{array} \left[\begin{array}{cccc|c} 1 & 2 & 2 & -1 & 5 \\ 0 & 1 & 2 & 1 & 2 \\ 0 & 3 & -4 & -1 & 0 \\ 1 & 2 & 1 & 3 & 0 \end{array}\right] \rightarrow \left\{ \begin{array}{rcl} x + 2y + 2z - w & = & 5 \\ y + 2z + w & = & 2 \\ -4z - w & = & 0 \\ 3w & = & 0 \end{array} \right.$$

$w = 0$

$z = 0$

$y = 2$

$x = 5 - 2y = 1$

$$
\begin{array}{c}
2 \\ 1 \\ 3 \\ 4
\end{array}
\left[\begin{array}{cccc|c}
1 & 2 & 2 & -1 & 5 \\
0 & 1 & 2 & 1 & 2 \\
0 & 3 & -4 & -1 & 0 \\
1 & 2 & 1 & 3 & 0
\end{array}\right]
\qquad
LU = \begin{bmatrix}
1 & 2 & 2 & -1 \\
0 & 1 & 2 & 1 \\
0 & 3 & 2 & 2 \\
1 & 4 & 2 & 3
\end{bmatrix}
$$

$$
U = \begin{bmatrix}
1 & 2 & 2 & -1 \\
 & 1 & 2 & 1 \\
 & & -4 & -1 \\
 & & & 3
\end{bmatrix}
\qquad
A = \begin{bmatrix}
0 & 1 & 2 & 1 \\
1 & 2 & 2 & -1 \\
0 & 3 & 2 & 2 \\
1 & 4 & 2 & 3
\end{bmatrix}
$$

$$
L = \begin{bmatrix}
1 & & & \\
0 & 1 & & \\
0 & 3 & 1 & \\
1 & 2 & 1 & 1
\end{bmatrix}
\qquad
P = \begin{pmatrix}
0 & 1 & & \\
1 & 0 & & \\
 & & 1 & \\
 & & & 1
\end{pmatrix}
$$

$$\boxed{PA = LU}$$

Come si usa la fattorizzazione LU di una matrice

$$Ax = b \Rightarrow PA = LU$$

1) moltiplico per P $\quad PAx = Pb$

2) uso fattorizzazione $\quad LUx = Pb$

3) Risolvo 2 sistemi in cascata

$$Lz = Pb$$

$$Ux = z$$

# Quanto costa fattorizzare

passo 1 $n-1$ elementi da azzerare

$(n-1)(n-1)$ moltiplicazioni

passo 2 $n-2$ elementi da azzerare

$(n-2)(n-2)$ moltiplicazioni

⋮

passo $n-1$ 1 elemento da azzerare

1 moltiplicazione

Totale

$$1^2 + 2^2 + 3^2 + \dots + (n-1)^2 = \frac{1}{3} n^3 + \frac{1}{2} n^2 + \frac{1}{6} n$$

$$\simeq \frac{n^3}{3}$$

Quanto costa risolvere un sistema in forma triangolare

$(\triangle)(x) = (b) \qquad (\nabla)(x) = (b)$

costo $1 + 2 + 3 + \quad + n$ prodotti o divisioni

$$= \frac{n(n+1)}{2} \approx \frac{n^2}{2}$$

quindi 2 sistemi triangolari costano circa $n^2$ operazioni

mentre la fattorizzazione costa $\frac{n^3}{3}$

$Ax = b_1 \qquad Ax = b_2 \; \ldots\ldots \; Ax = b_n$

# Matrici Frobenious

$$\begin{bmatrix} 1 & & & \\ -1/3 & 1 & & \\ -2/3 & & 1 & \\ -4/3 & & & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 & 0 & 0 \\ 1 & 1 & 2 & 0 \\ 2 & 1 & 1 & 2 \\ 4 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 0 & 0 \\ 0 & 1/3 & 2 & 0 \\ 0 & 7/3 & 1 & 2 \\ 0 & 11/3 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} a & b & c & d \end{bmatrix} \begin{array}{c} \\ \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix} \end{array} \begin{array}{l} a \\ b \\ c \\ d \end{array} = \begin{bmatrix} a + 5b + 9c + 13d \ldots \end{bmatrix}$$

# Matrici di Frobenius (bis)

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & -\frac{2}{3} & 1 & \\ & -\frac{1}{3} & & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 3 & 1 & 2 \\ 0 & 2 & 3 & 4 \\ 0 & 1 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 3 & 1 & 2 \\ 0 & 0 & 7/3 & 8/3 \\ 0 & 0 & 14/3 & 16/3 \end{bmatrix}$$

$$\left[\begin{array}{c|c} I & 0 \\ \hline 0 & L \end{array}\right] \left[\begin{array}{c|c} A & B \\ \hline 0 & C \end{array}\right] = \left[\begin{array}{c|c} A & B \\ \hline 0 & LC \end{array}\right]$$

$$L = \begin{pmatrix} 1 & & \\ \blacksquare & \ddots & \\ \blacksquare & & 1 \end{pmatrix} \qquad C = \begin{pmatrix} \bullet \\ \vdots \end{pmatrix}$$

(matrici a blocchi)

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} 14 & 20 \\ 32 & 44 \\ 6 & 8 \end{bmatrix}$$

$3 \times 3 \qquad 3 \times 2$

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 5 & 6 \\ \hline 1 & 1 & 1 \end{array}\right] \left[\begin{array}{cc} 1 & 1 \\ 2 & 2 \\ \hline 3 & 5 \end{array}\right] = \begin{bmatrix} \begin{pmatrix} 1 & 2 \\ 4 & 5 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} + \begin{pmatrix} 3 \\ 6 \end{pmatrix}\begin{pmatrix} 3 & 5 \end{pmatrix} \\ \begin{pmatrix} 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} + (1)\begin{pmatrix} 3 & 5 \end{pmatrix} \end{bmatrix}$$

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} E \\ F \end{bmatrix} = \begin{bmatrix} AE + BF \\ CE + DF \end{bmatrix}$$

Matrici di Frobenius come somma Identità + matrice rango 1

$$
\begin{bmatrix} 1 & & & \\ & 1 & & \\ & a & 1 & \\ & b & & 1 \end{bmatrix}
=
\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}
+
\begin{bmatrix} 0 & & & \\ & 0 & & \\ & a & 0 & \\ & b & & 0 \end{bmatrix}
$$

$$
= \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}
+ \underset{4\times 1}{\begin{pmatrix} 0 \\ 0 \\ a \\ b \end{pmatrix}} \underset{1\times 4}{\begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix}}
$$

$$
= \boxed{I + v e_2^T} \qquad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}
$$

# Inversa matrice di Frobenius

$$F = I + v e_k^T \qquad v = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ v_{k+1} \\ \vdots \\ v_n \end{pmatrix}$$

$$G = I - v e_k^T$$

$$FG = (I + v e_k^T)(I - v e_k^T)$$

$$= I + \cancel{v e_k^T} - \cancel{v e_k^T} - \underbrace{v}_{n\times 1} \underbrace{(\underbrace{e_k^T}_{1\times n} \underbrace{v}_{n\times 1})}_{1\times 1} \underbrace{e_k^T}_{1\times n} = I$$

$= 0$

$$e_k^T v = (0 \cdots 0 \underset{k}{1} 0 \cdots 0) \begin{pmatrix} 0 \\ \vdots \\ 0 \\ v_{k+1} \\ v_n \end{pmatrix} = 0$$

# Prodotto matrici di Frobenius

$$F_1 = I + v e_k^T \qquad v_0 = v_1 = \dots = v_k = 0$$

$$F_2 = I + w e_j^T \qquad w_0 = w_1 = \dots = w_j = 0$$

$$F_1 F_2 = (I + v e_k^T)(I + w e_j^T)$$

$$= \underbrace{I + v e_k^T + w e_j^T}_{} + v\overbrace{(e_k^T w)}^{=0 \text{ se } k \leq j} e_j^T$$

$$\begin{bmatrix} 1 & & & \\ & \ddots & & \\ & v & \ddots & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & w & \ddots & \\ & & & 1 \end{bmatrix} = \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & v \; w & \ddots & \\ & & & 1 \end{bmatrix}$$

Matrice di scambio e matrice di Frobenius $PP=I$

$$PF = P(I + v e_n^T)$$
$$= P + P v e_n^T$$
$$= (I + P v \underline{e_n^T P})P$$

Se $P$ scambia righe diverse dalla $n$ esima allora $e_n^T P = e_n^T$

$$PF = I + (Pv)e_n^T$$