GAUSS + LU + FROBENIUS

$$\begin{cases} y + 2z + w = 2 \\ x + 2y + 2z - w = 5 \\ 3y + 2z + 2w = 6 \\ x + 4y + 2z + 3w = 9 \end{cases}$$

$$A\vec{x} = b$$

$$A = \begin{pmatrix} 0 & 1 & 2 & 1 \\ 1 & 2 & 2 & -1 \\ 0 & 3 & 2 & 2 \\ 1 & 4 & 2 & 3 \end{pmatrix} \quad b = \begin{pmatrix} 2 \\ 5 \\ 6 \\ 9 \end{pmatrix} \quad \vec{x} = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$$

Metodo di Gauss

$$L_3 S_3 L_2 S_2 L_1 S_1 A = U$$

Applicheremo l'algoritmo passo passo memorizzando tutte le matrici coinvolte

$$A = \begin{pmatrix} 0 & 1 & 2 & 1 \\ 1 & 2 & 2 & -1 \\ 0 & 3 & 2 & 2 \\ 1 & 4 & 2 & 3 \end{pmatrix}$$

scambio 1 e 2 riga

$$S_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad S_1 A = \begin{pmatrix} 1 & 2 & 2 & -1 \\ 0 & 1 & 2 & 1 \\ 0 & 3 & 2 & 2 \\ 1 & 4 & 2 & 3 \end{pmatrix}$$

$$L_1 = \begin{pmatrix} 1 & & & \\ 0 & 1 & & \\ 0 & & 1 & \\ -1 & & & 1 \end{pmatrix} \qquad L_1 S_1 A = \begin{pmatrix} 1 & 2 & 2 & -1 \\ 0 & 1 & 2 & 1 \\ 0 & 3 & 2 & 2 \\ 0 & 2 & 0 & 4 \end{pmatrix}$$

$$S_2 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \qquad S_2 L_1 S_1 A =$$

$$L_2 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & -3 & 1 & \\ & -2 & & 1 \end{pmatrix} \qquad L_2 S_2 L_1 S_1 A = \begin{pmatrix} 1 & 2 & 2 & -1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & -4 & -1 \\ 0 & 0 & -4 & 2 \end{pmatrix}$$

$$L_2 S_2 L_1 S_1 A = \begin{pmatrix} 1 & 2 & 2 & -1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & -4 & -1 \\ 0 & 0 & -4 & 2 \end{pmatrix}$$

Non è necessario ma scambio riga 3 e 4 per far vedere effetto della permutazione

$$S_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad S_3 L_2 S_2 L_1 S_1 A = \begin{pmatrix} 1 & 2 & 2 & -1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & -4 & 2 \\ 0 & 0 & -4 & -1 \end{pmatrix}$$

$$L_3 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & -1 & 1 \end{pmatrix} \qquad L_3 S_3 L_2 S_2 L_1 S_1 A = \begin{pmatrix} 1 & 2 & 2 & -1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & -4 & 2 \\ 0 & 0 & 0 & -3 \end{pmatrix}$$

$$S_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad L_1 = \begin{pmatrix} 1 & & & \\ 0 & 1 & & \\ 0 & & 1 & \\ -1 & & & 1 \end{pmatrix} \quad S_2 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \quad L_2 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & -3 & 1 & \\ & -2 & & 1 \end{pmatrix}$$

$$S_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad L_3 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & -1 & 1 \end{pmatrix} \quad \underbrace{L_3 S_3 L_2 S_2 L_1 S_1 A}_{L^{-1}PA} = \begin{pmatrix} 1 & 2 & 2 & -1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & -4 & 2 \\ 0 & 0 & 0 & -3 \end{pmatrix} = U$$

$$L_3 S_3 L_2 L_1 S_1 A \qquad S_2 = I$$

$$S_3 L_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -2 & 0 & 1 \\ 0 & -3 & 1 & 0 \end{pmatrix} \times \underbrace{\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 0 & 1 \\ & & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 0 & 1 \\ & & 1 & 0 \end{pmatrix}}_{=I}$$

$$= \begin{pmatrix} 1 & & & \\ & 1 & & \\ & -2 & 1 & \\ & -3 & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 0 & 1 \\ & & 1 & 0 \end{pmatrix} = \tilde{L}_2 S_3$$

$$L_3 \tilde{L}_2 \underbrace{S_3 L_1} S_1 A$$

$$S_3 L_1 = \begin{pmatrix}1&&&\\&1&&\\&&0&1\\&&1&0\end{pmatrix}\begin{pmatrix}1&&&\\0&1&&\\0&&1&\\-1&&&1\end{pmatrix} = \begin{pmatrix}1&0&0&0\\0&1&0&0\\-1&0&0&1\\0&0&1&0\end{pmatrix} \overbrace{S_3 S_3}^{I}$$

$$= \underbrace{\begin{pmatrix}1&0&0&0\\0&1&0&0\\-1&0&0&1\\0&0&1&0\end{pmatrix}\begin{pmatrix}1&&&\\&1&&\\&&0&1\\&&1&0\end{pmatrix}}\begin{pmatrix}1&&&\\&1&&\\&&0&1\\&&1&0\end{pmatrix}$$

$$= \begin{pmatrix}1&0&0&0\\0&1&0&0\\-1&0&1&0\\0&0&0&1\end{pmatrix} S_3 = \tilde{L}_1 S_3$$

$$L_3 \tilde{L}_2 \tilde{L}_1 S_3 S_1 A = U$$

$$L_3 \tilde{L}_2 \tilde{L}_1 S_3 S_1 A = U$$

$$S_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad S_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \boxed{\begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix}} \end{pmatrix} \quad \tilde{L}_2 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & -2 & 1 & \\ & -3 & & 1 \end{pmatrix} \quad \tilde{L}_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$L_3 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & -1 & 1 \end{pmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & -2 & 1 & 0 \\ 0 & -3 & 0 & 1 \end{bmatrix}$$

$$P = S_3 S_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$L^{-1} = L_3 \tilde{L}_2 \tilde{L}_1 = \underbrace{\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & & & \\ & 1 & & \\ & -2 & 1 & \\ & -3 & & 1 \end{pmatrix}}_{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & -5 & -1 & 1 \end{pmatrix}} \begin{pmatrix} 1 & & & \\ 0 & 1 & & \\ -1 & & 1 & \\ 0 & & & 1 \end{pmatrix}$$

$$L = \tilde{L}_1^{-1} \tilde{L}_2^{-1} L_3^{-1}$$

$$= \begin{pmatrix} 1 & & & \\ 0 & 1 & & \\ 1 & & 1 & \\ 0 & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & \\ & 1 & & \\ & 2 & 1 & \\ & 3 & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & & & \\ 0 & 1 & & \\ 1 & 2 & 1 & \\ 0 & 3 & 1 & 1 \end{pmatrix}$$

$$PA = LU$$

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 2 & 1 \\ 1 & 2 & 2 & -1 \\ 0 & 3 & 2 & 2 \\ 1 & 4 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & & & \\ 0 & 1 & & \\ 1 & 2 & 1 & \\ 0 & 3 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 2 & -1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & -4 & 2 \\ 0 & 0 & 0 & -3 \end{pmatrix}$$

memorizzati in A