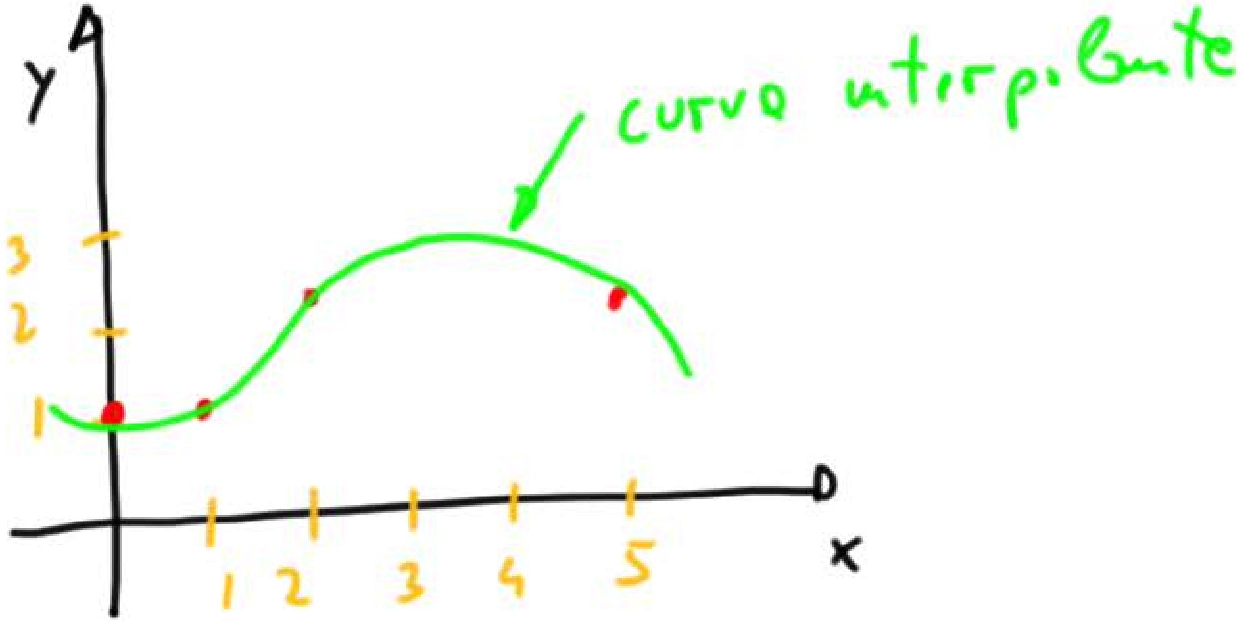
# INTERPOLAZIONE POLINOMIALE

| $x$ | $y$ |
|---|---|
| $x_0$ | $y_0$ |
| $x_1$ | $y_1$ |
| $\vdots$ | $\vdots$ |
| $x_n$ | $y_n$ |

esempio

| $x$ | $y$ |
|---|---|
| 0 | 1 |
| 1 | 1 |
| 5 | 2 |
| -3 | 1 |
| 2 | 3 |




cerco $p(x)$ tale che

$$p(x_0) = y_0$$
$$p(x_1) = y_1$$
$$\vdots$$
$$p(x_n) = y_n$$

ad esempio $p(x) = a_0 + a_1 x + a_2 x^2 + \quad + a_n x^n$

problema: trovare i coefficienti $a_n$ tali che siano valide

# FORMULAZIONE DEL PROBLEMA DI INTERPOLAZIONE

$$p(x) = a_0 + a_1 x + a_2 x^2 + \quad + a_n x^n$$

| $x$ | $y$ |
|---|---|
| $x_0$ | $y_0$ |
| $x_1$ | $y_1$ |
| $\vdots$ | $\vdots$ |
| $x_n$ | $y_n$ |

$$\begin{cases} p(x_0) = y_0 \\ p(x_1) = y_1 \\ \vdots \\ p(x_n) = y_n \end{cases} \Rightarrow \begin{cases} a_0 + a_1 x_0 + a_2 x_0^2 + \quad + a_n x_0^n = y_0 \\ a_0 + a_1 x_1 + a_2 x_1^2 + \quad + a_n x_1^n = y_1 \\ \vdots \\ a_0 + a_1 x_n + a_2 x_n^2 + \quad + a_n x_n^n = y_n \end{cases}$$

$$\begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ \vdots & & & & \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{bmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{pmatrix} \leftarrow \text{sistema lineare}$$

DOMANDA
esistenza
unicità
della soluzione

$$\begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^m \\ 1 & x_1 & x_1^2 & \cdots & x_1^m \\ \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^m \end{bmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_m \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{pmatrix} \qquad A a = y$$

$$A \qquad a \qquad y$$

$$A_{ij} = (x_{i-1})^{j-1} \qquad A \in \mathbb{R}^{(n+1)\times(m+1)}$$

$\Rightarrow n = m$ minimo richiesto

$\Rightarrow \det A \neq 0$

se $x_i = x_j \Rightarrow$ riga $i+1$ e $j+1$ esima coincidono $\Rightarrow A$ singolare

# DETERMINANTE VANDERMONDE

$$V(x_0 x_1 \ldots x_n) = \begin{vmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & & x_1^n \\ \vdots & & & & \\ 1 & x_n & x_n^2 & & x_n^n \end{vmatrix}$$

esempi $V(x_0 x_1) = \begin{vmatrix} 1 & x_0 \\ 1 & x_1 \end{vmatrix} = x_1 - x_0$

$$V(x_0, x_1 x_2) = \begin{vmatrix} 1 & x_0 & x_0^2 \\ 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \end{vmatrix} = (x_1 - x_0)(x_2 - x_0)(x_2 - x_1)$$

$$V(x_0 x_1 \quad x_n) = \prod_{j > i} (x_j - x_i)$$

schema / inizio dimostrazione

$$q(x) = V(x_0 x_1 \quad x_{n-1}, x) = x^n V(x_0 x_1 \quad x_{n-1}) + x^{n-1} \cdots$$

$$= (x - x_0)(x - x_1) \cdots (x - x_{n-1}) V(x_0 \ldots x_{n-1})$$

usando intelligentemente si prova

# DETERMINANTE VANDERMONDE

$$V(x_0\, x_1 \ldots x_n) = \begin{vmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & & x_1^n \\ \vdots & & & & \\ 1 & x_n & x_n^2 & & x_n^n \end{vmatrix}$$

$$V(x_0\, x_1 \ldots x_n) = \prod_{j>i} (x_j - x_i)$$

$\Rightarrow$

CONSEGUENZA:

IL DETERMINANTE È ZERO SOLO SE $x_i = x_j$ PER QUALCHE $i \neq j$

CIOÈ SE HO DUE PUNTI CON LA STESSA ASCISSA

ESEMPIO $\left(p(x) = 1 + x^2\right)$

| x | y |
|---|---|
| 0 | 1 |
| 1 | 2 |
| 2 | 5 |
| -1 | 2 |

$$\begin{cases} a_0 + a_1 0 + a_2 0^2 + a_3 0^3 = 1 \\ a_0 + a_1 1 + a_2 1^2 + a_3 1^3 = 2 \\ a_0 + a_1 2 + a_2 2^2 + a_3 2^3 = 5 \\ a_0 + a_1 (-1) + a_2 (-1)^2 + a_3 (-1)^3 = 2 \end{cases}$$

$$\begin{array}{cccc} a_0 & a_1 & a_2 & a_3 \end{array}$$
$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 2 \\ 1 & 2 & 4 & 8 & 5 \\ 1 & -1 & 1 & -1 & 2 \end{array}\right]$$

USANDO L'ALGORITMO DI GAUSS (FATTORIZZAZIONE LU) TROVIAMO LA SOLUZIONE

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & -1 & 1 & -1 \end{bmatrix}\begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & -1 & 1 & -1 \end{bmatrix}\begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 6 \\ 1 & -1 & 2 & 0 \end{bmatrix}\begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} \rightarrow$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 6 \\ 1 & -1 & 1 & -6 \end{bmatrix}\begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix}$$

$$A = LU \qquad P = I$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & -1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ 1 & 2 & 1 & \\ 1 & -1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ & 1 & 1 & 1 \\ & & 2 & 6 \\ & & & -6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ 1 & 2 & 1 & \\ 1 & -1 & 1 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ & 1 & 1 & 1 \\ & & 2 & 6 \\ & & & -6 \end{bmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}}_{z} = \begin{pmatrix} 1 \\ 2 \\ 5 \\ 2 \end{pmatrix}$$

$$Ax = b$$

$$PAx = Pb$$

$$L\underbrace{Ux}_{z} = Pb$$

$$\begin{cases} Lz = Pb \\ Ux = z \end{cases}$$

$$\begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ 1 & 2 & 1 & \\ 1 & -1 & 1 & 1 \end{bmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 5 \\ 2 \end{pmatrix} \Rightarrow$$

$$z_1 = 1$$

$$z_2 = 2 - z_1 = 2 - 1 = 1$$

$$z_3 = 5 - z_1 - 2z_2 = 5 - 1 - 2 = 2$$

$$z_4 = 2 - z_1 + z_2 - z_3 = 2 - 1 + 1 - 2 = 0$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ & 1 & 1 & 1 \\ & & 2 & 6 \\ & & & -6 \end{bmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 0 \end{pmatrix}$$

$$\alpha_3 = 0$$

$$\alpha_2 = 1$$

$$\alpha_1 = 1 - \alpha_2 - \alpha_3 = 0$$

$$\alpha_0 = 1 \Rightarrow 1 + x^2$$