DIFFERENZE DIVISE

Interpolazione di Newton

x	y
x_0	$f(x_0)$
x_1	$f(x_1)$
$\vdots$	
x_n	$f(x_n)$

$$p_0(x) = a_0 \omega_0(x) \qquad \omega_0(x) = 1$$

$$p_1(x) = a_0 \omega_0(x) + a_1 \omega_1(x) \qquad \omega_1(x) = x - x_0$$

$$\vdots$$

$$p_k(x) = a_0 \omega_0(x) + a_1 \omega_1(x) + \dots + a_k \omega_k(x)$$

$$\omega_k(x) = (x - x_0) \cdots (x - x_{k-1})$$

a_0 dipende da x_0 (e dalla funzione f)

a_1 dipende da x_0, x_1

$\vdots$

a_k dipende da $x_0, x_1, \dots, x_k$

$$a_0 = f[x_0] \qquad a_n = f[x_0 x_1 \; x_n]$$

$$a_1 = f[x_0 x_1]$$

DIFFERENZE DIVISE

DIFFERENZE DIVISE (PROPRIETÀ)

x	y
x_0	$f(x_0)$
x_1	$f(x_1)$
x_n	$f(x_n)$

x	y
x_{σ_0}	$f(x_{\sigma_0})$
x_{σ_1}	$f(x_{\sigma_1})$
x_{σ_n}	$f(\sigma_n)$

$$p_n(x) = p_n(x)$$

σ è una permutazione di $\{0,1,\dots,n\}$

Esempio

$$\begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix} \xrightarrow{\sigma} \begin{pmatrix} 3 \\ 2 \\ 0 \\ 1 \end{pmatrix} \qquad \begin{aligned} \sigma_0 &= 3 \\ \sigma_1 &= 2 \\ \sigma_2 &= 0 \\ \sigma_3 &= 1 \end{aligned}$$

I POLINOMI SONO IDENTICI IN QUANTO INTERPOLANO GLI STESSI PUNTI!

$$p_n(x) = a_0[x_0]\,\omega_0(x) + \dots + a_n[x_0 x_1 \dots x_n]\,\omega_n(x)$$

$$p_n(x) = a_0[x_{\sigma_0}]\,\omega_0(x) + \dots + a_n[x_{\sigma_0} x_{\sigma_1} \dots x_{\sigma_n}]\,\omega_n(x)$$

$$\omega_n(x) = (x - x_0)(x - x_1)\dots(x - x_{n-1}) = x^n + x^{n-1}(\dots)$$

$$\omega_n(x) = (x - x_{\sigma_0})(x - x_{\sigma_1})\dots(x - x_{\sigma_{n-1}}) = x^n + x^{n-1}(\dots)$$

$$\Rightarrow a_n[x_0 \dots x_n] = a_n[x_{\sigma_0} x_{\sigma_1} \dots x_{\sigma_n}]$$

LE DIFFERENZE DIVISE SONO FUNZIONI SIMMETRICHE

$f[x_0 x_1 \ x_n] = f[x_{\sigma_0} x_{\sigma_1} \dots x_{\sigma_n}]$ (per ogni permutazione σ)

Esempio di funzione simmetrica

$g(x,y) = xy + x + y + 1$

$g(x,y,z) = xyz$ $\quad g(x\, y\, z) = xy + yz + zx - 1$

Esempio di funzioni NON simmetriche

$g(x,y) = x + 2y$ $\quad g(x,y,z) = xy + z$

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$f[x_0] = f(x_0)$ $\quad f[x_0, x_1] = \dfrac{f(x_0) - f(x_1)}{x_0 - x_1}$

$p(x) = f[x_0] + (x - x_0) f[x_0, x_1]$

$f[x_0 x_1 x_2] = ??$

DERIVAZIONE DELLA RICORSIONE

X	Y
x_0	$f(x_0)$
x_1	$f(x_1)$
x_{n-1}	$f(x_{n-1})$
x_n	$f(x_n)$

X	Y
x_0	$f(x_0)$
x_1	$f(x_1)$
x_n	$f(x_n)$
x_{n-1}	$f(x_{n-1})$

$\omega_0(x) = 1 = \omega_0(x)$

$\omega_1(x) = x - x_0 = \omega_1(x)$

$\omega_n(x) = \omega_{n-1}(x)(x - x_{n-1})$

$\omega_n(x) = \omega_{n-1}(x)(x - x_n)$

$\omega_{n-1}(x) = (x - x_0)(x - x_1) \dots (x - x_{n-2}) = \omega_{n-1}(x)$

$$p(x) = f[x_0]\omega_0(x) + f[x_0 x_1]\omega_1(x) + \dots + f[x_0 \; x_{n-2}, x_{n-1}]\omega_{n-1}(x) + f[x_0 \dots x_n]\omega_n(x)$$

$$p(x) = f[x_0]\omega_0(x) + f[x_0 x_1]\omega_1(x) + \dots + f[x_0 \; x_{n-2}, x_n]\omega_{n-1}(x) + f[x_0 \; x_{n-2}, x_n, x_{n-1}]\omega_n(x)$$

$$f[x_0 \; x_n] = f[x_0 \; x_{n-2}, x_n, x_{n-1}]$$

$$0 = p(x) - p(x) = f[x_0 \dots x_{n-2}, x_{n-1}]\omega_{n-1}(x) - f[x_0 \; x_{n-2}, x_n]\omega_{n-1}(x) + f[x_0 \; x_n](\omega_n(x) - \omega_n(x))$$

$$0 = f[x_0 x_1 \dots x_{n-2}, x_{n-1}]\,\underline{\omega_{n-1}(x)}$$
$$- f[x_0 x_1 \quad x_{n-2}, x_n]\,\underline{\omega_{n-1}(x)}$$
$$+ f[x_0 \quad x_n]\,\underline{\omega_{n-1}(x)}\,(\cancel{x} - x_{n-1} - (\cancel{x} - x_n)\ )$$

$$f[x_0 - x_{n-2}, x_{n-1}] - f[x_0 \quad x_{n-2}, x_n]$$
$$= f[x_0 \quad x_n](x_{n-1} - x_n)$$

$$\boxed{f[x_0 \quad x_n] = \frac{f[x_0 \quad x_{n-2}, x_{n-1}] - f[x_0 - x_{n-2}, x_n]}{x_{n-1} - x_n}}$$

$$f[\underbrace{x_0 - x_{n-2}}, \alpha, \beta] = \frac{f[\overbrace{x_0 - x_{n-2}}, \alpha] - f[\overbrace{x_0 - x_{n-2}}, \beta]}{\alpha - \beta}$$

$$f[\underbrace{x_0 - x_{n-2}}, \alpha, \beta] = \frac{f[\overbrace{x_0 - x_{n-2}}, \alpha] - f[\overbrace{x_0 - x_{n-2}}, \beta]}{\alpha - \beta}$$

Usando la simmetria delle differenze divise

$$f[\underbrace{\alpha, \overbrace{\boxed{\quad}}}, \beta] = \frac{f[\overbrace{\alpha, \boxed{\quad}}] - f[\overbrace{\boxed{\quad}, \beta}]}{\alpha - \beta}$$

Esempio

$$
\begin{aligned}
&f(x_0) \\
&f(x_1) \quad f[\overset{\alpha}{x_0}\,\boxed{x_1}] = \frac{f[x_0] - f[x_1]}{x_0 - x_1} \\
& \qquad\qquad\qquad\qquad\qquad\qquad \frac{f[x_0\,x_1] - f[x_1\,x_2]}{x_0 - x_2} \\
&f(x_2) \quad f[\boxed{x_1}\,\underset{\beta}{x_2}] = \frac{f[x_1] - f[x_2]}{x_1 - x_2}
\end{aligned}
$$

x	y				
x_0	$f[x_0]$				
		$f[x_0,x_1]$			
x_1	$f[x_1]$		$f[x_0\,x_1\,x_2]$		
		$f[x_1,x_2]$		$f[x_0\,x_1\,x_2\,x_3]$	
x_2	$f[x_2]$		$f[x_1\,x_2\,x_3]$		$f[\text{—}]$
		$f[x_2\,x_3]$		$f[x_1\,x_2\,x_3\,x_4]$	
x_3	$f[x_3]$		$f[x_2\,x_3\,x_4]$		
		$f[x_3\,x_4]$			
x_4	$f[x_4]$				

$$p(x) = f[x_0]\,\omega_0(x) + f[x_0\,x_1]\,\omega_1(x) + f[x_0\,x_1,x_2]\,\omega_2(x)$$
$$f[x_0\,x_1\,x_2\,x_3]\,\omega_3(x) + f[x_0\,x_1\,x_2\,x_3\,x_4]\,\omega_4(x)$$

ESEMPIO USO DIFFERENZE DIVISE (soluzione $1+x^3-x^4$)

x	$f(x)$
0	1
1	1
2	-7
-1	-1
-2	-23

$$\frac{f[x_0]-f[x_1]}{x_0-x_1} = \frac{1-1}{0-1} = 0$$

$$\frac{f[x_1]-f[x_2]}{x_1-x_2} = \frac{1-(-7)}{1-2} = -8$$

$$\frac{-7-(-1)}{2-(-1)} = \frac{-6}{3} = -2$$

$$\frac{-1-(-23)}{-1-(-2)} = \frac{22}{1} = 22$$

$$\frac{0-(-8)}{0-2} = -4$$

$$\frac{-8-(-2)}{1-(-1)} = -3$$

$$\frac{-2-22}{2-(-2)} = -6$$

$$\frac{-4-(-3)}{0-(-1)} = -1$$

$$\frac{-3-(-6)}{1-(-2)} = 1$$

$$\frac{-1-1}{0-(-2)} = -1$$

$$p(x) = 1\,\omega_0(x) + 0\,\omega_1(x) - 4\,\omega_2(x) - 1\,\omega_3(x) - 1\,\omega_4(x)$$

x	$f(x)$
0	1
1	1
2	-7
-1	-1
-2	-23

$$p(x) = 1\,\omega_0(x) + 0\,\omega_1(x) - 4\,\omega_2(x) - 1\,\omega_3(x) - 1\,\omega_4(x)$$

$$\omega_0(x) = 1$$

$$\omega_1(x) = x$$

$$\omega_2(x) = x(x-1) = x^2 - x$$

$$\omega_3(x) = (x^2 - x)(x-2) = x^3 - 3x^2 + 2x$$

$$\omega_4(x) = (x^3 - 3x^2 + 2x)(x+1) = x^4 - 2x^3 - x^2 + 2x$$

$$p(x) = 1 + \\ -4(x^2 - x) \\ -(x^3 - 3x^2 + 2x) \\ -(x^4 - 2x^3 - x^2 + 2x)$$

$$1 + x^3 - x^4$$

$\boxed{\text{PROVATE A CALCOLARLO CON LAGRANGE}}$

ALTRO ESEMPIO (soluzione $\frac{1}{2} - \frac{x^3}{3}$)

$$\omega_0(x) = 1$$
$$\omega_1(x) = x$$
$$\omega_2(x) = x(x-2) = x^2 - 2x$$
$$\omega_3(x) = (x^2-2x)(x+1)$$
$$= x^3 - x^2 + 2x$$

x	$f(x)$
0	$\frac{1}{2}$
2	$-\frac{13}{6}$
-1	$\frac{5}{6}$
1	$\frac{1}{6}$

$$\frac{\frac{1}{2} + \frac{13}{6}}{0-2} = -\frac{4}{3}$$

$$\frac{-\frac{13}{6} - \frac{5}{6}}{2-(-1)} = -1$$

$$\frac{\frac{5}{6} - \frac{1}{6}}{-1-1} = -\frac{1}{3}$$

$$-\frac{1}{3}$$

$$-\frac{2}{3}$$

$$\frac{-\frac{1}{3} + \frac{2}{3}}{0-1} = -\frac{1}{3}$$

$$p(x) = \frac{1}{2}\,1 - \frac{8}{6}x - \frac{1}{3}(x^2-2x) - \frac{1}{3}(x^3 - x^2 + 2x)$$

$$= \frac{1}{2} + x\,\cancel{\left(-\frac{4}{3} + \frac{2}{3} + \frac{1}{3}\right)} + x^2\cancel{\left(-\frac{1}{3} + \frac{1}{3}\right)} - \frac{1}{3}x^3 = \frac{1}{2} - \frac{x^3}{3}$$

STIMA ERRORE INTERPOLAZIONE

$f(x)$ funzione da approssimare

$p(x)$

x	x_0	x_1	$\dots$	x_n
$f(x)$	$f(x_0)$	$f(x_1)$	$\dots$	$f(x_n)$

$f(x) - p(x) = E(x)$ = errore di interpolazione

posso stimare $E(x)$?

$E(x_k) = 0 \quad k = 0, 1, \dots, n$ perché $f(x_k) = p(x_k)$

$E(x) = c(x)(x - x_0)(x - x_1) \cdots (x - x_n) = c(x)\,\omega_{n+1}(x)$

$E(x)$?

$$E(x) = f(x) - p(x) = e(x)\,\omega_{n+1}(x)$$

$$g(x) = f(x) - p(x) - e(z)\,\omega_{n+1}(x)$$

proprietà della funzione $g(x)$

$$g(x_k) = \underbrace{f(x_k) - p(x_k)}_{0} - e(z)\underbrace{\omega_{n+1}(x_k)}_{=0} = 0$$

$$g(z) = f(z) - p(z) - e(z)\,\omega_{n+1}(z) = 0 \quad \left[\begin{array}{l}\text{per definizione} \\ E(x)\end{array}\right]$$

$\boxed{\text{$g(x)$ è funzione che si annulla in $n+2$ punti}}$

Sia $h(x)$ funzione che si annulla in 2 punti $x_0\ x_1$

$\Rightarrow$ Rolle $\exists\, c \in (x_0\ x_1)$ $h'(c)=0$

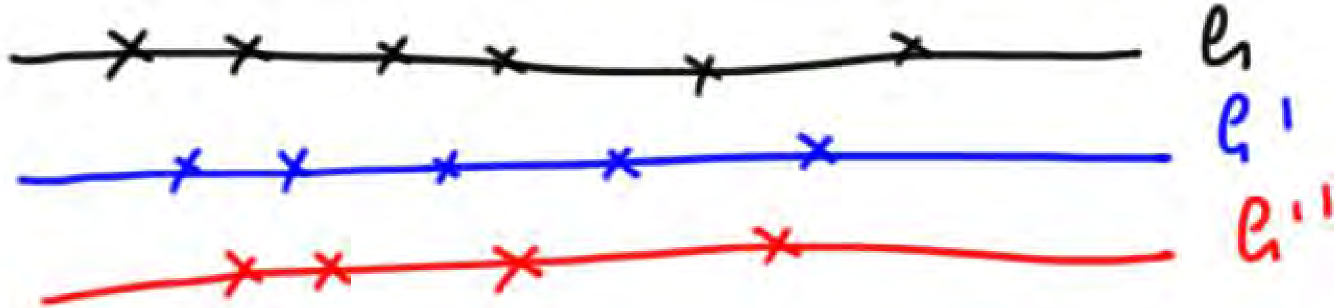
Sia $h(x)$ funzione che si annulla in 3 punti $x_0\ x_1\ x_2$ (ordinati e distinti)

$\Rightarrow$ Rolle^2 $\exists\, c_1 \in (x_0\ x_1)$ $h'(c_1)=0$

$\exists\, c_2 \in (x_1\ x_2)$ $h'(c_2)=0$

$h'(c_1) = h'(c_2) = 0 \Rightarrow \exists\, c_3 \in (c_1\, c_2) \subseteq (x_0\ x_2)$

$h''(c_3)=0$



Se $h(x)$ si annulla in $n+1$ punti distinti $\exists\, c \in (x_0\ x_n)$ (se ordinati)

$h^{(n)}(c)=0$

$$g(x) = f(x) - p(x) - e(z)\,\omega_{n+1}(x) = 0$$

$\boxed{g(x) \text{ è funzione che si annulla in } n+2 \text{ punti}}$

Usando Rolle^{n+1} $\qquad \omega_{n+1}(x) = x^{n+1} + \dots$

$$g^{(n+1)}(c) = 0 \qquad c \in I(x_0 \dots x_n, z) = (\min\{x_0 \dots x_n, z\}, \max\{x_0 \dots x_n, z\})$$

$$g^{(n+1)}(c) = f^{(n+1)}(c) - \underbrace{p^{(n+1)}(c)}_{=0} - e(z)\,\omega_{n+1}^{(n+1)}(x)$$

$$= f^{(n+1)}(c) - e(z)(n+1)! \Rightarrow$$

ESERCIZIO
TROVARE
FORMULA ERRORE