EQUAZIONI DIFFERENZIALI ORDINARIE (ODE)

$$(*)\begin{cases} y' = f(x,y) \\ \underbrace{y(a) = y_a}_{\text{dato iniziale}} \end{cases}$$

$y' = f(x,y)$

$y'(x) = f(x, y(x))$

TROVARE UNA FUNZIONE $y(x)$ tale che $(*)$ sia soddisfatta

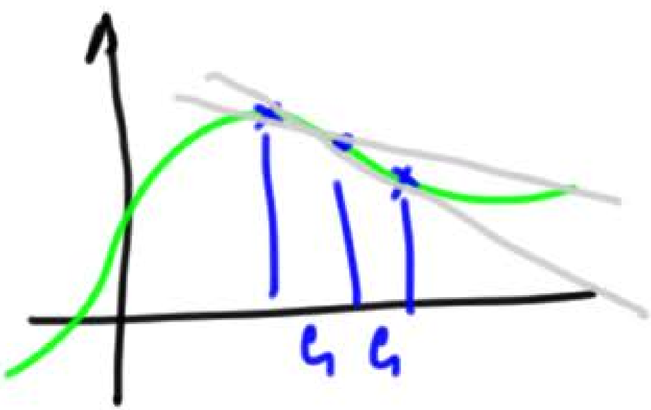
Teorema (PEANO) se $f(x,y)$ è CONTINUA allora esiste una soluzione

Teorema: Se $f(x,y)$ è CONTINUA e Lipschitz in y (cioè $|f(x,y) - f(x,z)| \leq L|y-z|$) allora esiste una unica soluzione del problema $(*)$

PER APPROSSIMARE LE SOLUZIONI DI UNA ODE SERVE PRIMA CAPIRE COME APPROSSIMARE DELLE DERIVATE:

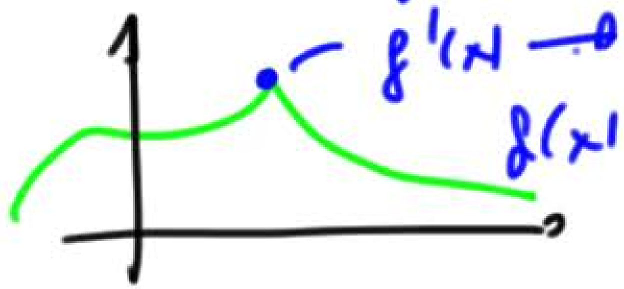
$$y'(x) = \lim_{h \to 0} \frac{y(x+h) - y(x)}{h} = \lim_{h \to 0} \frac{y(x) - y(x-h)}{h}$$

$$= \lim_{h \to 0} \frac{y(x+h) - y(x-h)}{2h}$$



Definizioni equivalenti a meno di spigoli

$f'(x) \to$ dipende dalla definizione



Possiamo approssimare la derivata con DIFFERENZE FINITE

$$f'(x) \approx \frac{f(x+h) - f(x)}{h} \qquad h = \text{finito}$$

VARI MODI PER APPROSSIMARE $f'(x)$

MODO 1 (USO POLINOMIO INTERPOLANTE)

$$\left.\begin{array}{l} x,\ x+h,\ x+2h,\ x-h,\ x-2h,\ \dots \\ f(x),\ f(x+h),\ f(x+2h)\ \dots \end{array}\right\}$$ costruisco polinomio interpolante

$p(z)$ tale che $p(x) = f(x)$ $\quad p(x+h) = f(x+h)$
$p(x-h) = f(x-h)$.

$$\boxed{f'(x) \approx p'(x)}$$

Esempio $\{x, x-h\} \Rightarrow p(z) = f(x) + (z-x)\, f[x, x-h]$

$$p'(z) = f[x, x-h] = \frac{f(x) - f(x-h)}{x-(x-h)}$$

$$= \frac{f(x) - f(x-h)}{h}$$

Esempio $\{x-h, x, x+h\}$

$$p'(x) = \frac{f(x+h) - f(x-h)}{2h}$$

Tutto ok ma non ho idea dell'errore commesso

PROBLEMA

dato $$f'(x) \approx \frac{f(x+h) - f(x)}{h}$$ vorrei scriverlo come

$$f'(x) = \frac{f(x+h) - f(x)}{h} + E$$

E = qualcosa di calcolabile / stimabile

MODO 2

$$f'(x) = A f(x) + B f(x+h) + \text{ERRORE}$$

CERCO A e B in modo che ERRORE sia "piccolo"

Come al solito si usa TAYLOR

$$f'(x) = A f(x) + B f(x+h) + \text{ERRORE}$$

$$\text{ERRORE} = f'(x) - A f(x) - B f(x+h)$$

$$= f'(x) - A f(x) - B\left(f(x) + f'(x) h + f''(\xi) \frac{h^2}{2} \right)$$

$$= f(x)\underbrace{(-A-B)}_{0} + f'(x)\underbrace{(1-Bh)}_{0} - B f''(\xi)\frac{h^2}{2}$$

$$\begin{cases} A + B = 0 \\ 1 - Bh = 0 \end{cases} \qquad B = \frac{1}{h} \quad A = -\frac{1}{h}$$

$$\text{ERRORE} = -f''(\xi)\frac{h}{2}$$

$$\boxed{f'(x) = \frac{f(x+h) - f(x)}{h} - \overbrace{f''(\xi)\frac{h}{2}}^{\text{ERRORE}}}$$

SEMPRE MODO 2

$$f'(x) = A f(x) + B f(x+h) + C f(x-h) + \text{ERRORE}$$

$$\text{ERRORE} = f'(x) - A f(x) - B f(x+h) - C f(x-h)$$

$$= f'(x) - A f(x) - B\left(f(x) + f'(x) h + f''(x)\frac{h^2}{2} + f'''(x)\frac{h^3}{6} + O(h^4)\right)$$

$$- C\left(f(x) - f'(x) h + f''(x)\frac{h^2}{2} - f'''(x)\frac{h^3}{6} + O(h^4)\right)$$

$$= f(x)(-A - B - C) + f'(x)(1 - Bh + Ch)$$

$$+ f''(x)\frac{h^2}{2}(-B - C) + f'''(x)\frac{h^3}{6}(C - B)$$

$$+ O(h^4)(B + C)$$

$$\left.\begin{cases} A + B + C = 0 \\ 1 - Bh + Ch = 0 \\ B + C = 0 \end{cases}\right\} \quad A = 0,\quad B = \frac{1}{2h},\quad C = -\frac{1}{2h}$$

$$\text{ERRORE} = f'''(x)\frac{h^2}{6} + O(h^3)$$

Quindi

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h} + f'''(x)\frac{h^2}{6} + O(h^3)$$

FACENDO IL CONTO IN MANIERA UN PO DIVERSA

ERRORE: $f'(x) - A f(x) - B\left(f(x) + f'(x)h + f''(x)\frac{h^2}{2} + f'''(\xi)\frac{h^3}{6}\right)$
$- C\left(f(x) - f'(x)h + f''(x)\frac{h^2}{2} - f'''(\omega)\frac{h^3}{6}\right)$

$$= f(x)(-A-B-C) + f'(x)(1 - Bh - Ch)$$
$$f''(x)\frac{h^2}{2}(B+C) - \frac{h^3}{6}\left(B f'''(\xi) - C f'''(\omega)\right)$$

$\Rightarrow$ $A = 0 \quad B = \frac{1}{2h} \quad C = -\frac{1}{2h}$ ERRORE $= -\frac{h^2}{6}\left(\frac{f'''(\xi) + f'''(\omega)}{2}\right) = f'''(\alpha)$

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h} + f'''(\alpha)\frac{h^2}{6} \qquad \alpha \in (x-h, x+h)$$

IN GENERALE POSSIAMO USARE LA PROCEDURA

① CON POLINOMIO INTERPOLANTE TROVO FORMULA A DIFFERENZE FINITE

② CON TAYLOR CALCOLO ERRORE (LOCALE)

ESEMPIO

$$f'(x) \approx \frac{f(x) - f(x-h)}{h}$$

$$\frac{f(x) - f(x-h)}{h} = \frac{\cancel{f(x)} - \left(\cancel{f(x)} - h f'(x) + \frac{h^2}{2} f''(\xi)\right)}{h} =$$

$$= f'(x) - \frac{h}{2} f''(\xi)$$

ESERCIZIO

$$f'(x) = \frac{f(x+h) - f(x)}{h} + E \qquad E\,?$$

$$f''(x) = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} + E \qquad E\,?$$

USARE TAYLOR

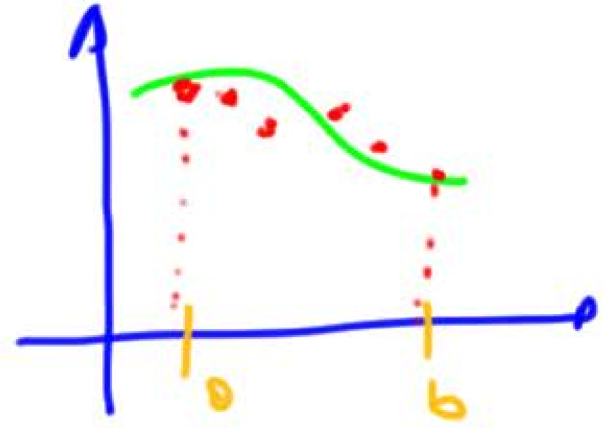
USARE POLY INTERPOLANTE PER TROVARE DIFFERENZA FINITA

$\{x, x+h, x+2h\}$ trova $f'(x) = \,?\, + \text{ERRORE}$

$f''(x) = \,?\, + \text{ERRORE}$

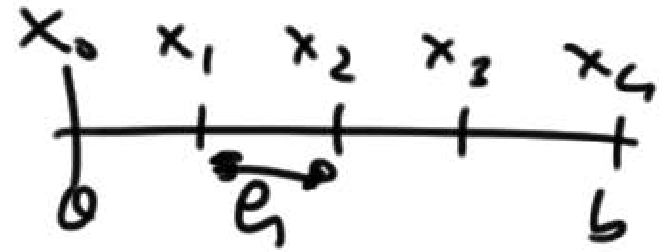
METODO DI EULERO

$$\begin{cases} y' = f(x,y) \\ y(a) = y_a \end{cases} \qquad x \in [a,b]$$



$$h = \frac{b-a}{n}$$

$$x_0 = a$$
$$x_1 = a + h$$
$$x_2 = a + 2h$$
$$\vdots$$
$$x_k = a + kh$$
$$\vdots$$
$$x_n = a + nh = a + (b-a) = b$$



$$y'(x) = f(x, y(x)) \qquad y'(x_k) = f(x_k, y(x_k))$$

$$\Updownarrow$$

$$\frac{y(x_k+h) - y(x_k)}{h} + \text{ERRORE}$$

$$y'(x_k) = f(x_k, y(x_k))$$

$$\frac{y(x_k+h) - y(x_k)}{h} - \underbrace{y''(\xi_k)\frac{h}{2}}_{\text{ERRORE}} = f(x_k, y(x_k))$$

$$x_k + h = x_{k+1}$$

$$\frac{y(x_{k+1}) - y(x_k)}{h} - y''(\xi_k)\frac{h}{2} = f(x_k, y(x_k))$$

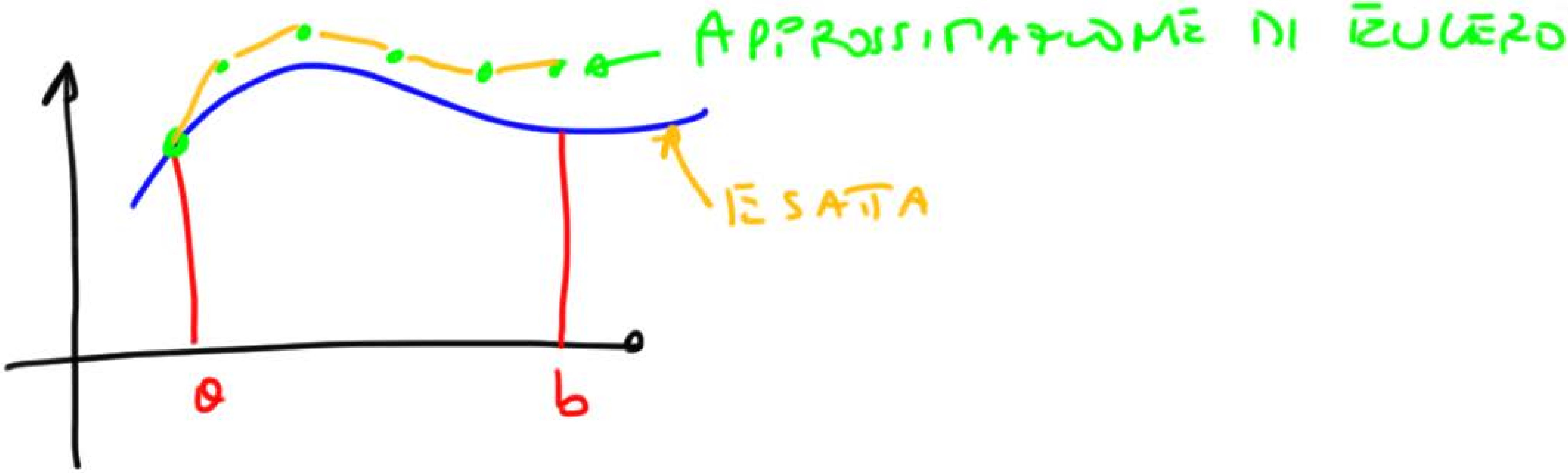
Chiamo $y_k \simeq y(x_k)$ approssimazione soluzione esatta

$$\frac{y_{k+1} - y_k}{h} = f(x_k, y_k) \Rightarrow y_{k+1} = y_k + h f(x_k, y_k)$$

Metodo di Eulero

$$\begin{cases} y' = f(x,y) & x \in [a,b) \\ y(a) = y_a & h = \dfrac{b-a}{n} \end{cases}$$

$$\begin{cases} y_0 = y_a \\ y_{k+1} = y_k + h\, f(x_k, y_k) \end{cases} \qquad x_k = a + kh$$



STUDIO DELLA CONVERGENZA

$$\begin{cases} y' = f(x,y) \\ y(a) = \alpha \end{cases} \qquad h = \frac{b-a}{n} \qquad x_k = a + kh \qquad y_k \simeq y(x_k)$$

DEF ERRORE $\varepsilon_k = y(x_k) - y_k$

$$y(x_{k+1}) = y(x_k) + h\, f(x_k, y(x_k)) + y''(\xi_n)\frac{h^2}{2}$$

$$y_{k+1} = y_k + h\, f(x_k, y_k)$$

$$\varepsilon_{k+1} = \varepsilon_k + h\left[f(x_k, y(x_k)) - f(x_k, y_k)\right] + y''(\xi_n)\frac{h^2}{2}$$

Se $f(x,y)$ è derivabile in y

$$f(x_k, y(x_k)) - f(x_k, y_k) = \frac{\partial f}{\partial y}(x_k, z_k)\underbrace{(y(x_k) - y_k)}_{\varepsilon_k}$$

$$\varepsilon_{k+1} = \varepsilon_k + h\frac{\partial f}{\partial y}(x_k, z_k)\,\varepsilon_k + y''(\xi_k)\frac{h^2}{2}$$

$$|\varepsilon_{k+1}| \le |\varepsilon_k| + h\,|\varepsilon_k|\,\left|\frac{\partial f}{\partial y}(x_k, z_k)\right| + |y''(\xi_k)|\,\frac{h^2}{2}$$

$$M = \sup_{\substack{x \in [a,b] \\ z \in \mathbb{R}}} \left|\frac{\partial f}{\partial y}(x, z)\right| < \infty$$

(Forte, si può indebolire complicando la dimostrazione)

$$N = \sup_{x \in [a,b]} |y''(x)| < \infty \quad \text{se la soluzione è } C^2$$

$$|\varepsilon_{k+1}| \le |\varepsilon_k| + Mh\,|\varepsilon_k| + N\frac{h^2}{2} \qquad \Big[\ \varepsilon_0 = 0$$

$$= (1 + Mh)\,|\varepsilon_k| + N\frac{h^2}{2}$$

$$|\varepsilon_{k+1}| \le \underbrace{(1+Mh)}_{A}\, |\varepsilon_k| + \underbrace{N\frac{h^2}{2}}_{B} \quad \text{com} \quad |\varepsilon_0| = 0$$

$$\begin{cases} |\varepsilon_0| = 0 \\ |\varepsilon_{k+1}| \le A|\varepsilon_k| + B \end{cases}$$

$$|\varepsilon_1| \le A|\varepsilon_0| + B = B$$

$$|\varepsilon_2| \le A|\varepsilon_1| + B \le AB + B$$

$$|\varepsilon_3| \le A|\varepsilon_2| + B \le A(AB+B) + B = A^2B + AB + B$$

$$= (A^2 + A + 1)B$$

$$|\varepsilon_4| \le A|\varepsilon_3| + B \le A(A^2+A+1)B + B$$

$$= (A^3 + A^2 + A + 1)B$$

$$|\varepsilon_{k+1}| \le (A^k + A^{k-1} + \quad + A + 1)B$$

$$|\varepsilon_{k+1}| \le \underbrace{(1+Mh)}_{A}\,|\varepsilon_k| + \underbrace{N\frac{h^2}{2}}_{B} \quad \text{con} \quad |\varepsilon_0| = 0$$

$$|\varepsilon_{k+1}| \le (\underbrace{A^k + A^{k-1} + \quad + A}_{\text{SERIE GEOMETRICA}} + 1)\,B$$

SERIE GEOMETRICA

$$1 + q + q^2 + \quad + q^n = \frac{1-q^{n+1}}{1-q}$$

$$|\varepsilon_{k+1}| \le \frac{1-A^{k+1}}{1-A}\,B = \frac{1-(1+Mh)^{k+1}}{1-(1+Mh)}\,N\frac{h^2}{2}$$

$$= \frac{1-(1+Mh)^{k+1}}{M\cancel{h}}\,N\frac{h^{\cancel{2}}}{2} = \frac{N}{M}\frac{h}{2}\left((1+Mh)^{k+1} - 1\right)$$

$$\le \frac{N}{M}\frac{h}{2}(1+Mh)^{k+1}$$

$$|\varepsilon_{k+1}| \le \underbrace{(1+Mh)}_{A}\, |\varepsilon_k| + \underbrace{N \frac{h^2}{2}}_{B} \quad \text{con} \quad |\varepsilon_0| = 0$$

$$|\varepsilon_k| \le \frac{N}{M}\,\frac{h}{2}\,(1+Mh)^k \qquad k = 0, 1, \dots, n$$

$$e^x = 1 + x + \frac{z^2}{2} \qquad 0 \le z \le x \qquad (\text{per } x \ge 0$$

$$e^x \ge 1 + x \qquad e^{Mh} \ge 1 + Mh$$

$$e^{Mhk} = (e^{Mh})^k \ge (1+Mh)^k$$

Metodo di Eulero Converge

$\Downarrow$

$$|\varepsilon_k| \le \frac{N}{M}\,\frac{h}{2}\,e^{Mhk} \le \frac{N}{M}\,\frac{h}{2}\,e^{M(b-a)} = Ch$$