RIASSUNTO METODI NUMERICI PER ODE

BASATI SULLA SERIE DI TAYLOR

$$\begin{cases} y' = f(x, y) \\ y(a) = y_a \end{cases}$$

$y(x)$ soluzione esatta
soddisfa $y'(x) = f(x, y(x))$

cerchiamo la soluzione nell'intervallo $[a, b]$
e ci accontentiamo di approssimarla
in punti discreti x_k dove $x_k = a + kh$
$h = (b-a)/n$

$$y(x_{k+1}) = y(x_k + h) = y(x_k) + h\,y'(x_k) + \frac{h^2}{2}\,y''(x_k) + \dots + \frac{h^p}{p!}\,y^{(p)}(x_k) + \underbrace{\frac{h^{p+1}}{(p+1)!}\,y^{(p+1)}(\xi_k)}_{\text{TRASCURARE ERROR}}$$

[scrivere $y^{(r)}(x_k)$ in funzione di $f(x, y)$

TRASCURARE ERROR

FORMULA DI AVANZAMENTO $\boxed{y_k \approx y(x_k)}$

$$y_{k+1} = y_k + h\,y'_k + \frac{h^2}{2} y''_k + \dots + \frac{h^p}{p!} y^{(p)}_k$$

Espressioni da determinare

Se $y(x)$ è soluzione esatta

$\boxed{y'(x) = f(x, y(x))}$ Relazione [illegible]

$$y'_k = f(x_k, y_k)$$

$$y''(x) = \frac{d}{dx} f(x, y(x)) = \frac{\partial f}{\partial x}(x, y(x)) + \frac{\partial f}{\partial y}(x, y(x))\, f(x, y(x))$$

$$y''_k = \frac{\partial f}{\partial x}(x_k, y_k) + \frac{\partial f}{\partial y}(x_k, y_k)\, f(x_k, y_k)$$

si può procedere in questo modo derivando ad ogni passo per calcolare le derivate successive $y^{(r)}_k$

FORMULA RICORRENTE (per $D_r(x,y)$)

Definendo $y_k^{(r)} = D_r(x_k, y_k)$ il metodo basato su Taylor si scrive

$$y_{k+1} = y_k + h \underbrace{D_1(x_k, y_k)}_{y'_k} + \frac{h^2}{2} \underbrace{D_2(x_k, y_k)}_{y''_k} + \dots + \frac{h^p}{p!} \underbrace{D_p(x_k\, y_k)}_{y^{(p)}_k}$$

$$D_1(x,y) = f(x,y)$$

$$y^{(r+1)}(x) = \frac{d}{dx} y^{(r)}(x) = \frac{d}{dx} D_r(x, y(x)) =$$

$$= \frac{\partial D_r}{\partial x}(x, y(x)) + \frac{\partial D_r}{\partial y}(x, y(x))\, y'(x)$$

$$D_{r+1}(x, y(x)) = \frac{\partial D_r}{\partial x}(x, y(x)) + \frac{\partial D_r}{\partial y}(x, y(x))\, f(x, y(x))$$

$$D_{r+1}(x,y) = \frac{\partial D_r}{\partial x}(x,y) + \frac{\partial D_r}{\partial y}(x,y)\, f(x,y)$$

ESEMPIO

SCRIVERE METODO BASATO SU TAYLOR ORDINE 3 PER LA SEGUENTE ODE

$$\begin{cases} y' = x^2 y + e^{-x} \\ y(0) = 1 \end{cases}$$

$$D_1(x,y) = x^2 y + e^{-x}$$

$$D_2(x,y) = \partial_x D_1(x,y) + \partial_y D_1(x,y)\, f(x,y)$$
$$= 2xy - e^{-x} + x^2 (x^2 y + e^{-x}) = y(2x + x^4) + e^{-x}(x^2 - 1)$$

$$D_3(x,y) = \partial_x D_2(x,y) + \partial_y D_2(x,y)\, f(x,y)$$
$$= y(2 + 4x^3) - e^{-x}(x^2 - 1) + e^{-x} 2x + (2x + x^4)(x^2 y + e^{-x})$$

$$y_{k+1} = y_k + D_1(x_k, y_k)\, h + D_2(x_k, y_k) \frac{h^2}{2}$$
$$+ D_3(x_k, y_k) \frac{h^3}{3}$$

ERRORE METODO BASATO SU TAYLOR

$y(x_k)$ = soluzione esatta calcolata in x_k

y_k = " approssimata "

$$y(x_{k+1}) = y(x_k) + y'(x_k)h + \quad + y^{(p)}(x_k)\frac{h^p}{p!} + y^{(p+1)}(\xi_k)\frac{h^{p+1}}{(p+1)!}$$

$$y_{k+1} = y_k + y'_k h + \quad + y^{(p)}_k \frac{h^p}{p!}$$

$$\varepsilon_k = y(x_k) - y_k$$

$$\begin{aligned}\varepsilon_{k+1} = \varepsilon_k &+ h\left(y'(x_k) - y'_k\right) + \\ &+ \frac{h^2}{2}\left(y''(x_k) - y''_k\right) \\ &\vdots \\ &+ \frac{h^p}{p!}\left(y^{(p)}(x_k) - y^{(p)}_k\right) + y^{(p+1)}(\xi_k)\frac{h^{p+1}}{(p+1)!}\end{aligned}$$

$$\varepsilon_{k+1} = \varepsilon_k + h\left(y'(x_k) - y'_k\right) + \frac{h^2}{2}\left(y''(x_k) - y''_k\right)$$

$$\vdots$$

$$+ \frac{h^p}{p!}\left(y^{(p)}(x_k) - y^{(p)}_k\right) + y^{(p+1)}(\xi_k)\,\frac{h^{p+1}}{(p+1)!}$$

x ottimizzazione

$$y'(x_k) - y'_k = y'(x_k) - D_1(x_k, y_k)$$

$$= \underbrace{f(x_k, y(x_k))}_{\text{x ODE}} - f(x_k, y_k)$$

$z_k \in I(y_k, y(x_k))$

Teorema Lagrange Analisi 1

$$= \frac{\partial f}{\partial x}(x_k, z_k)\left(y(x_k) - y_k\right)$$

$$= \frac{\partial f}{\partial x}(x_k, z_k)\,\varepsilon_k$$

$$\varepsilon_{k+1} = \varepsilon_k + h\left(y'(x_k) - y'_k\right) + \frac{h^2}{2}\left(y''(x_k) - y''_k\right) + \vdots + \frac{h^p}{p!}\left(y^{(p)}(x_k) - y^{(p)}_k\right) + y^{(p+1)}(\xi_k)\frac{h^{p+1}}{(p+1)!}$$

$$D_2(x,y) = \partial_x D_1 + \partial_y D_1 f = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f$$

$$y''(x_n) - D_2(x_k, y_n) = y''(x_k) - \frac{\partial f}{\partial x}(x_k, y_n) - \frac{\partial f}{\partial y}(x_n, y_n) f(x_n, y_n)$$

$$= \frac{\partial f}{\partial x}(x_k, y(x_k)) + \frac{\partial f}{\partial y}(x_k, y(x_k)) f(x_k, y(x_k))$$

$$- \frac{\partial f}{\partial x}(x_n, y_n) - \frac{\partial f}{\partial y}(x_n, y_n) f(x_n, y_n)$$

$$= \frac{\partial f}{\partial x}(x_n, z_n)\underbrace{\left(y(x_n) - y_n\right)}_{\varepsilon_n} + \cdots\cdots$$

si trova $\boxed{}\ \varepsilon_n$

ma le espressioni sono complicate

PER STIMARE ERRORE CONVIENE DEFINIRE
ERRORE LOCALE

IN PRATICA E' ERRORE CHE SI OTTIENE SE

$y_k = y(x_k)$ $\qquad y^{(r)}_k = D_k(x_k, y_k) = y^{(r)}(x_k)$

$$y(x_{k+1}) = y(x_k) + y'(x_k)h + \qquad + y^{(p)}(x_k)\frac{h^p}{p!} + \text{Errore}$$

$$\quad \| \qquad\quad \| \qquad\qquad\qquad \|$$

$$y_{k+1} = y_k + y'_k h + + \qquad + y^{(p)}_k \frac{h^p}{p!}$$

$\tau_{k+1} = \text{ERRORE} = \dfrac{h^{p+1}}{(p+1)!} y^{(p+1)}(\xi_n)$

$\|$

$=$ errore ε_{k+1} nel caso $y_k = y(x_k)$

PICCOLA ASTRAZIONE

Metodo basato su Taylor

$$y_{k+1} = y_k + h\, y'_k + \frac{h^2}{2} y''_k + \quad + \frac{h^p}{p!} y^{(p)}_k$$

$$= \underline{\phi}(x_k, y_k, h)$$

Funzione descrivente il metodo numerico

soluzione approssimata a partire da $y(x_k)$ esatta

$$\overline{y}_{k+1} = \phi(x_k, y(x_k), h)$$

$$\underbrace{\overline{y}_{k+1} - y(x_{k+1})}_{-\tau_{k+1}} = \phi(x_k, y(x_k), h) - y(x_{k+1})$$

Resto di Taylor

$$y(x_{k+1}) - \phi(x_k, y(x_k), h) = \tau_{k+1} = \underbrace{\frac{h^{p+1}}{(p+1)!} y^{(p+1)}(\xi_k)}$$

DEFINIZIONE

Dato metodo numerico per approssimare ODE

$$\begin{cases} y' = f(x,y) \\ y(a) = y_0 \end{cases}$$

$[a,b]$ in passo ampiezza h

$x_k = a + kh$

$$\begin{cases} y_0 = y_a \\ y_{k+1} = \phi(x_k, y_k, h) \end{cases}$$

GENERICO METODO NUMERICO

[AD 1 PASSO ESPLICITO]

$y(x_{k+1}) - \phi(x_k, y(x_k), h) = \tau_{k+1}$ = Errore Locale

$\dfrac{\tau_{k+1}}{h}$ = ERRORE LOCALE TRONCAMENTO

Ricorrenze per ERRORE

$$y_{k+1} = \phi(x_k, y_k, h)$$

$$y(x_{k+1}) = \phi(x_k, y(x_k), h) + \tau_{k+1}$$ [x definizione di τ_{k+1}]

$$y(x_{k+1}) - y_{k+1} = \left[\phi(x_k, y(x_k), h) - \phi(x_k, y_k, h)\right] + \tau_{k+1}$$

$$\varepsilon_{k+1} = \frac{\partial \phi}{\partial y}(x_k, z_k, h)\,(\underbrace{y(x_k) - y_k}_{\varepsilon_k}) + \tau_{k+1}$$

$$\boxed{\varepsilon_{k+1} = \frac{\partial \phi}{\partial y}(x_k, z_k, h)\,\varepsilon_k + \tau_{k+1}}$$

$$z_k \in I(y_k, y(x_k))$$

Ricorrenza errore

In generale dato metodo numerico

$y_{k+1} = \phi(x_k, y_k, h)$ con errore locale

$\tau_{k+1} = y(x_{k+1}) - \phi(x_k, y(x_k), h)$

$\varepsilon_{k+1} = \frac{\partial \phi}{\partial y}(x_k, \eta_k, h)\, \varepsilon_k + \tau_{k+1}$

$|\varepsilon_{k+1}| \le M |\varepsilon_k| + N$

dove $M = \sup \left| \frac{\partial \phi}{\partial y} \right|$ $\quad N = \sup |\tau|$

In generale $N = C h^{p+1}$

Adottando la stima dell'errore fatta per metodo di Eulero otteniamo

$|\varepsilon_k| \le \tilde{C} h^p$ (Esercizio da fare a casa, da soli....)

Runge-Kutta ....

Dato metodo numerico

$$y_{k+1} = \phi(x_k, y_k, h)$$

quello che conta (per ora) è l'errore locale

$$\tau_{k+1} = y(x_{k+1}) - \phi(x_k, y_k, h)$$

che nel caso del metodo basato su Taylor vale

$$\tau_{k+1} = \frac{h^{p+1}}{(p+1)!} y^{(p+1)}(\xi_k) \quad \text{cioè} \quad \tau_{k+1} = O(h^{p+1})$$

$$\text{cioè} \quad |\tau_{k+1}| \leq C h^{p+1}$$

Domanda (retorica) si può ottenere $\tau_{k+1} = O(h^{p+1})$ senza usare derivate parziali di $f(x,y)$?

Esempio ordine 2

TAYLOR

$$y_{n+1} = y_n + y'_n h + y''_n \frac{h^2}{2}$$

$$= y_n + D_1(x_n, y_n) h + D_2(x_n, y_n) \frac{h^2}{2}$$

$$= y_n + f(x_n, y_n) h + \left[\partial_x f + \partial_y f \, f\right] \frac{h^2}{2}$$

RK

$$y_{n+1} = y_n + A f(x_n, y_n) + B\, f(x_n + C, y_n + D)$$

$$= \Phi(x_n, y_n, h)$$

CALCOLO A, B, C, D in modo che τ_{n+1}

sia il più "piccolo possibile"

$$y_{k+1} = y_k + A f(x_k, y_k) + B f(x_k + C, y_k + D)$$

$$\tau_{k+1} = y(x_{k+1}) - \left[y(x_k) + A f(x_k, y(x_k)) + B f(x_k + C, y(x_k) + D) \right]$$

$$= y(x_k) + h\, y'(x_k) + \frac{h^2}{2} y''(x_k) + O(h^3)$$

$$- \Big[y(x_k) + A f(x_k, y(x_k)) + B \Big[f(x_k, y(x_k)) + C \frac{\partial f}{\partial x}(x_k, y(x_k))$$

$$+ D \frac{\partial f}{\partial y}(x_k, y(x_k)) +$$

$$\frac{1}{2} \frac{\partial^2 f}{\partial x^2}(x_k + z, y_k + w)\, C^2 +$$

$$\frac{1}{2} \frac{\partial^2 f}{\partial y^2}(x_k + z, y_k + w)\, D^2 +$$

$$\frac{\partial^2 f}{\partial x \partial y}(x_k + z, y_k + w)\, CD \Big] \Big]$$

$$y'(x_k) = f(x_k, y(x_k))$$

$$y''(x_k) = \cdots$$

Semplificazione (notazione)

$$f = f(x_k, y(x_k))$$

$$f_x = \partial_x f(x_k, y(x_k))$$

$$f_y = \partial_y f(x_k, y(x_k)).$$

$$\tau_{k+1} = y(x_{k+1}) - [y(x_k) + A f(x_k, y(x_k)) + B f(x_k + C, y(x_k) + D)]$$

$$= y(x_k) + h y'(x_k) + \frac{h^2}{2} y''(x_k) + O(h^3)$$

$$- \Big[y(x_k) + A f(x_k, y(x_k)) + B \Big[f(x_k, y(x_k)) + C \frac{\partial f}{\partial x}(x_k, y(x_k)) + D \frac{\partial f}{\partial y}(x_k, y(x_k)) +$$

$$\frac{1}{2} \frac{\partial^2 f}{\partial x^2}(x_k + z, y_k + w) C^2 + \frac{1}{2} \frac{\partial^2 f}{\partial y^2}(x_k + z, y_k + w) D^2 + \frac{\partial^2 f}{\partial x \partial y}(x_k + z, y_k + w) CD \Big] \Big]$$

$$y'(x_k) = f(x_k, y(x_k))$$

$$y''(x_k) = \dots$$

$$\tau_{k+1} = h f + \frac{h^2}{2}(f_x + f_y f) + O(h^2)$$

$$- \Big[A f + B \Big[f + C f_x + D f_y + \frac{1}{2} \tilde{f}_{xx} C^2 + \frac{1}{2} \tilde{f}_{yy} D^2 + \tilde{f}_{xy} CD \Big] \Big]$$

$$= f[h - A - B] + f_x \left(\frac{h^2}{2} - BC \right) + f_y \left(\frac{h^2}{2} f - BD \right) \dots$$

$$\Rightarrow \tau_{k+1} = O(h^p) \qquad p \text{ più alto possibile} \dots$$