METODO di RUNGE KUTTA

$$\begin{cases} y' = f(x,y) \\ y(0) = y_0 \end{cases}$$

$$f(x+\alpha, y+\beta) = f(x,y) + \alpha \frac{\partial f}{\partial x}(x,y) + \beta \frac{\partial f}{\partial y}(x,y) + \dots$$

$h = \frac{b-a}{n}$ $\qquad x_n = a + hk$

$$y(x_{n+1}) = y(x_n) + \underbrace{y'(x_n)}_{f(x_n, y(x_n))} h + \underbrace{y''(x_n)}_{\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f\right)} \frac{h^2}{2} + \underline{O(h^3)}$$

$$y_{n+1} = y_n + h\left[\alpha f(x_n, y_n) + \beta f(x_n + \gamma h, y_n + \omega h)\right]$$

$$y_{n+1} = \Phi(x_n, y_n, h)$$

$$\Phi(x,y,h) = y + h\left[\alpha f(x,y) + \beta (x + \gamma h, y + \omega h)\right]$$

Metodo basato su Taylor ordine 2

$$\phi^T(x,y,h) = y + h f(x,y) + \frac{h^2}{2}\left[\frac{\partial f}{\partial x}(x,y) + \frac{\partial f}{\partial y}(x,y)\, f(x,y)\right]$$

Metodo da determinare

$$\phi(x,y,h) = y + h\left[\alpha f(x,y) + \beta f(x+\gamma h, y+\omega h)\right]$$

$$y(x_{n+1}) - \phi^T(x_n, y(x_n), h) = \tau = O(h^3)$$

[Sviluppando la soluzione esatta con Taylor si verifica]

$$\overbrace{y(x_{n+1}) - \phi(x_n, y(x_n), h)}^{\tau_{n+1}} =$$

$$\overbrace{\cancel{y(x_n)} + h\, y'(x_n) + \frac{h^2}{2} y''(x_n) + O(h^3)}$$

$$- \cancel{y(x_n)} - h\left[\alpha f(x_n, y(x_n)) + \beta f(x_n+\gamma h, y(x_n)+\omega h)\right]$$

$$= h f(x_n, y(x_n)) + \frac{h^2}{2}\left[\frac{\partial f}{\partial x}(x_n, y(x_n)) + \frac{\partial f}{\partial y}(x_n, y(x_n))\, f(x_n, y(x_n))\right]$$

$$+ O(h^3) - h\left[\alpha f(x_n, y(x_n)) + \beta f(x_n+\gamma h, y(x_n)+\omega h)\right]$$

$$\tau_{n+1} = y(x_{n+1}) - \psi(x_n, y(x_n), h) =$$

$$= h f(x_n, y(x_n)) + \frac{h^2}{2}\left[\frac{\partial f}{\partial x}(x_n, y(x_n)) + \frac{\partial f}{\partial y}(x_n, y(x_n))\, f(x_n, y(x_n))\right] + O(h^3)$$

$$- h\Big\{\alpha f(x_n, y(x_n)) + \beta\Big[f(x_n, y(x_n)) + \gamma h \frac{\partial f}{\partial x}(x_n, y(x_n)) + \omega h \frac{\partial f}{\partial y}(x_n, y(x_n)) +$$

$$\frac{1}{2}(\gamma h)^2 \frac{\partial^2 f}{\partial x^2}(x_n+\theta, y(x_n)+T) +$$

$$O(h^2) \Leftarrow \quad \frac{1}{2}(\omega h)^2 \frac{\partial^2 f}{\partial y^2}(x_n+\theta, y(x_n)+T) +$$

$$\gamma\omega h^2 \frac{\partial^2 f}{\partial x \partial y}(x_n+\theta, y(x_n)+T)\Big]\Big\}$$

$$\rightarrow \quad 1-\alpha-\beta = 0$$

$$\frac{1}{2} - \beta\gamma = 0$$

$$\frac{f}{2} - \beta\omega = 0$$

$$\tau_{n+1} = h f(x_n, y(x_n))(1-\alpha-\beta) \leftarrow O(h)$$

$$+ h^2 \frac{\partial f}{\partial x}(x_n, y(x_n))\left(\frac{1}{2} - \beta\gamma\right) \leftarrow O(h^2)$$

$$O(h^2)$$

$$+ h^2 \frac{\partial f}{\partial y}(x_n, y(x_n))\left(\frac{f(x_n, y(x_n))}{2} - \beta\omega\right) + O(h^3)$$

$$\tau_{n+1} = h\, f(x_n, y(x_n))\,(1-\alpha-\beta)$$
$$+ h^2 \frac{\partial f}{\partial x}(x_n, y(x_n))\left(\frac{1}{2} - \beta\gamma\right)$$
$$+ h^2 \frac{\partial f}{\partial y}(x_n, y(x_n))\left(\frac{f(x_n, y(x_n))}{2} - \beta\omega\right) + O(h^3)$$

$$\begin{cases} 1-\alpha-\beta = 0 \\ \frac{1}{2} - \beta\gamma = 0 \\ \frac{f(x_n, y(x_n))}{2} - \beta\omega = 0 \end{cases} \qquad \begin{cases} \alpha+\beta = 1 \\ \beta\gamma = \frac{1}{2} \\ \beta\tilde{\omega} = \frac{1}{2} \end{cases}$$

$$\omega = f(x_n, y(x_n))\,\tilde{\omega}$$

$\alpha = 0$, $\beta = 1$, $\gamma = \frac{1}{2}$, $\tilde{\omega} = \frac{1}{2}$

$\alpha = \frac{1}{2}$, $\beta = \frac{1}{2}$, $\gamma = 1$, $\tilde{\omega} = 1$

Ci sono infinite soluzioni ...

$$\phi(x,y,h) = y + h\left[\alpha f(x,y) + \beta f(x+\gamma h,\ y+\tilde{\omega} h)\right]$$

$\alpha = 0$, $\beta = 1$, $\gamma = \frac{1}{2}$, $\tilde{\omega} = \frac{1}{2}$

$\alpha = \frac{1}{2}$, $\beta = \frac{1}{2}$, $\gamma = 1$, $\tilde{\omega} = 1$

Soluzione 1

$$y_{n+1} = y_n + h f\Big(x_n + \frac{h}{2},\ \underbrace{y_n + \frac{h}{2} f(x_n, y_n)}\Big)$$

$$\begin{cases} y_{n+\frac{1}{2}} = y_n + \frac{h}{2} f(x_n, y_n) \\ y_{n+1} = y_n + h f\big(x_{n+\frac{1}{2}},\ y_{n+\frac{1}{2}}\big) \end{cases}$$

Metodo Collatz o Eulero Modificato

$$x_{n+\frac{1}{2}} = a + \left(n+\tfrac{1}{2}\right)h = x_n + \frac{h}{2}$$

Soluzione 2

$$y_{n+1} = y_n + h\Big(\frac{1}{2} f(x_n, y_n) + \frac{1}{2} f\big(x_n + h,\ \underbrace{y_n + h f(x_n, y_n)}\big)\Big)$$

Metodo di Heun

$$\begin{cases} \tilde{y}_{n+1} = y_n + h f(x_n, y_n) \\ y_{n+1} = y_n + \frac{h}{2}\big(f(x_n, y_n) + f(x_{n+1}, \tilde{y}_{n+1})\big) \end{cases}$$

Metodi di Runge-Kutta

$$\begin{cases} y' = f(x,y) \\ y(a) = y_0 \end{cases}$$

$$y_{n+1} = y_n + b_1 K_1 + b_2 K_2 + \dots + b_s K_s$$

s = numero di stadi

$$\begin{cases} K_1 = h f\left(x_n + c_1 h,\ y_n + \sum_{j=1}^{s} a_{1j} K_j\right) \\ K_2 = h f\left(x_n + c_2 h,\ y_n + \sum_{j=1}^{s} a_{2j} K_j\right) \\ \vdots \\ K_s = h f\left(x_n + c_2 h,\ y_n + \sum_{j=1}^{s} a_{sj} K_j\right) \end{cases}$$

RK generale

Se $a_{ij} = 0$ per $j \geq i$ allora si scrive

$$\begin{cases} K_1 = h f(x_n + c_1 h,\ y_n) \\ K_2 = h f(x_n + c_2 h,\ y_n + a_{21} K_1) \\ K_3 = h f(x_n + c_3 h,\ y_n + a_{31} K_1 + a_{32} K_2) \\ \vdots \\ K_s = h f(x_n + c_s h,\ y_n + a_{s1} K_1 + \dots + a_{s\,s-1} K_{s-1}) \end{cases}$$

RK espliciti

Scrittura compatta

$$y_{k+1} = y_n + \sum_{j=1}^{s} b_j K_j \qquad K_j = h\, f\left(x_n + c_j h,\ y_n + \sum_{e=1}^{s} a_{je} K_e\right)$$

Tableau

$$\begin{array}{c|ccc} c_1 & a_{11} & & a_{1s} \\ c_2 & \vdots & & \\ & & & \\ c_s & a_{s1} & \cdots & a_{ss} \\ \hline & b_1\ b_2 & & b_s \end{array}$$

Esempio

$$\begin{array}{c|c} 0 & 0 \\ \hline & 1 \end{array}$$

$$\left.\begin{aligned} y_{k+1} &= y_n + K_1 \\ K_1 &= h\, f(x_n, y_n) \end{aligned}\right\} \to y_{k+1} = y_n + h\, f(x_n, y_n)$$

Eulero Esplicito

Esempio

$$\begin{array}{c|c} 1 & 1 \\ \hline & 1 \end{array}$$

$$\left[\begin{aligned} y_{k+1} &= y_n + K_1 \\ K_1 &= h\, f(x_n + h,\ y_n + K_1) = h\, f(x_{k+1}, y_{k+1}) \end{aligned}\right.$$

$$\Rightarrow y_{k+1} = y_n + h\, f(x_{k+1}, y_{k+1})$$

Eulero Implicito

Esempio

$$\begin{array}{c|cc} 0 & & \\ 1 & 1 & \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array}$$

$$y_{k+1} = y_k + \frac{1}{2}K_1 + \frac{1}{2}K_2$$

$$K_1 = h f(x_k, y_k)$$

$$K_2 = h f(x_k + h, y_k + K_1)$$

$$y_k + K_1 = y_k + h f(x_k, y_k)$$

$$\begin{cases} y_{k+1} = y_k + \frac{h}{2}\left(f(x_k, y_k) + f(x_{n+1}, \tilde{y}_{k+1})\right) \\ \tilde{y}_{n+1} = y_n + h f(x_n, y_n) \end{cases}$$

Metodo di Heun

ESERCIZIO TROVATE IL TABLEAU PER IL METODO DI COLLATZ

RK ORDINE 4

$$y_{k+1} = y_k + \frac{1}{6}k_1 + \frac{1}{3}k_2 + \frac{1}{3}k_3 + \frac{1}{6}k_4$$

$$k_1 = h f(x_k, y_k)$$

$$k_2 = h f\left(x_k + \frac{h}{2}, y_k + \frac{1}{2}k_1\right)$$

$$k_3 = h f\left(x_k + \frac{h}{2}, y_k + \frac{1}{2}k_2\right)$$

$$k_4 = h f(x_k + h, y_k + k_3)$$

NOTO

$$y_{k+1} = y_k + \frac{h}{6}k_1 + \frac{h}{3}k_2 \dots$$

$$k_1 = f(x_k, y_k)$$

$$k_2 = f\left(x_k + \frac{h}{2}, y_k + \frac{h}{2}k_1\right)$$

TABLEAU

$$\begin{array}{c|cccc}
0 & 0 & 0 & 0 & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\
\frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\
1 & 0 & 0 & 1 & 0 \\
\hline
 & \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6}
\end{array}$$

ESEMPIO

Metodo basato su Taylor ordine 4

$$\begin{cases} y' = xy \\ y(0) = 1 \end{cases}$$

$$D_1(x,y) = xy$$

$$D_2(x,y) = y + x(xy) = y(1+x^2)$$

$$D_3(x,y) = 2xy + (1+x^2)xy = 3xy + x^3y = (3+x^2)xy$$

$$D_3(x,y) = 3y + 3x^2y + x(3+x^2)xy$$
$$= y(3+3x^2+3x^2+x^4)$$

$$y_{n+1} = y_n + h\,D_1(x_n,y_n) + \frac{h}{2}D_2(x_n,y_n) + \frac{h^3}{6}D_3(x_n,y_n) + \frac{h^4}{24}D_4(\ )$$

$$\begin{array}{c|cc} 0 & \cdot & \cdot \\ 1 & \frac{1}{2} & \frac{1}{2} \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array}$$

$$y_{k+1} = y_k + \frac{1}{2}K_1 + \frac{1}{2}K_2$$

$$K_1 = h f(x_k, y_k) = h\,x_k y_k$$

$$K_2 = h f\left(x_k + h, y_k + \frac{K_1}{2} + \frac{K_2}{2}\right) =$$

$$= h\,x_{k+1}\left(y_k + \frac{K_1}{2} + \frac{K_2}{2}\right) = x_{k+1}\left(y_k + \frac{h}{2}x_k y_n + \frac{K_2}{2}\right)$$

$$
\begin{array}{c|cc}
0 & \cdot & \cdot \\
1 & \frac{1}{2} & \frac{1}{2} \\
\hline
 & \frac{1}{2} & \frac{1}{2}
\end{array}
$$

$$y_{k+1} = y_k + \frac{1}{2}K_1 + \frac{1}{2}K_2$$

$$K_1 = h\, f(x_k, y_k) = h\, x_k y_k$$

$$K_2 = h f\left(x_k + h,\ y_k + \frac{K_1}{2} + \frac{K_2}{2}\right) =$$

$$= h\, x_{k+1}\left(y_k + \frac{K_1}{2} + \frac{K_2}{2}\right) = h\, x_{k+1}\left(y_k + \frac{h}{2} x_k y_k + \frac{K_2}{2}\right)$$

$$K_2 = h\, x_{k+1} y_k + \frac{h^2}{2} x_k x_{k+1} y_k + h\, x_{k+1} \frac{K_2}{2}$$

$$K_2 = \frac{h\, x_{k+1} y_k + \frac{h^2}{2} x_k x_{k+1} y_k}{1 - \frac{h}{2} x_{k+1}}$$