Riassunto RK

$$\begin{cases} y' = f(x,y) \\ y(a) = y_a \end{cases}$$

$$\begin{array}{c|c} c & A \\ \hline & b^T \end{array}$$

TABLEAU

Avanzamento

$$y_{n+1} = y_n + \sum_{j=1}^{s} b_j K_j$$

$$K_j = h f\left(x_n + c_j h,\ y_n + \sum_{l=1}^{s} a_{jl} K_l\right)$$

s = numero di stadi

Se $a_{jl} = 0$ per $l \geq j$ allora il metodo è esplicito

Modo alternativo

$$y_{n+1} = y_n + h \sum_{j=1}^{s} b_j K_j$$

$$K_j = f\left(x_n + c_j h,\ y_n + h \sum_{l=1}^{s} a_{jl} K_l\right)$$

Altro modo

$$y_{n+1} = y_n + h \sum_{j=1}^{s} b_j f(\tilde{x}_j, \tilde{y}_j)$$

$$\tilde{x}_j = x_n + c_j h$$

$$\tilde{y}_j = y_n + h \sum_{l=1}^{s} a_{jl} f(\tilde{x}_l, \tilde{y}_l)$$

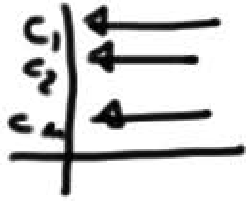
RELAZIONI ALGEBRICHE PER CALCOLARE ERRORE LOCALE

$y_{k+1} = \phi(x_k, y_k, h)$ $\qquad$ $y(x_{k+1}) - \phi(x_k, y_k, h) = \tau_k$

$|\tau_k| \le c\, h^p$? $\qquad$ $\mathbb{1} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$

CONDIZIONE 0 $\qquad \tau_k \to 0 \quad h \to 0$

$$\sum_{j=1}^{s} a_{ij} = c_i \quad i = 1, 2, \dots, s \qquad A\mathbb{1} = c$$



ORDINE 1 $\qquad |\tau_k| \le c\, h^2$

$$\sum_{j=1}^{s} b_i = 1 \qquad b \cdot \mathbb{1} = 1$$

ORDINE 2 $\qquad |\tau_k| \le c\, h^3$

$$\sum_{j=1}^{s} b_j c_j = \frac{1}{2} \qquad b \cdot c = \frac{1}{2}$$

ORDINE 3 $\quad |\tau_n| \le c h^4$

$$\sum_{j=1}^{s} b_j c_j^2 = \frac{1}{3} \qquad \sum_{i,j=1}^{s} b_i a_{ij} c_j = \frac{1}{6} \Rightarrow b^T A c = \frac{1}{6}$$

ORDINE 4 $\quad |\tau_n| \le c h^5$

$$\sum_{j=1}^{s} b_j c_j^3 = \frac{1}{4} \Rightarrow b \cdot c^3 = \frac{1}{4}$$

$$\sum_{i,j=1}^{s} b_i c_i a_{ij} c_j = \frac{1}{8} \qquad (bc) \cdot Ac = \frac{1}{8}$$

$$\sum_{i,j=1}^{s} b_i a_{ij} c_j^2 = \frac{1}{12} \qquad b \cdot (Ac^2) = \frac{1}{12}$$

$$\sum_{i,j,k=1}^{s} \overbrace{b_i a_{ij}}^{b^T A} \overbrace{a_{jk} c_k}^{Ac} = \frac{1}{24} \qquad (A^T b) \cdot (Ac) = \frac{1}{24}$$

ESEMPIO (RALSTON METHOD)

$$\begin{array}{c|cc} 0 & \cdot & \cdot \\ 2/3 & 2/3 & \cdot \\ \hline & 1/4 & 3/4 \end{array} \leftarrow 0 \quad \text{METODO ESPLICITO}$$

CONSISTENZA

$c_1 = 0 = a_{11} + a_{12} = 0 + 0 = 0$ (OK)

$c_2 = 2/3 = a_{21} + a_{22} = 2/3 + 0 = 2/3$ (OK)

ORDINE 1

$b_1 + b_2 = 1/4 + 3/4 = 1$ (OK)

ORDINE 2

$b \cdot c = \begin{pmatrix} 1/4 \\ 3/4 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 2/3 \end{pmatrix} = 1/2$ (OK)

ORDINE = 2

ORDINE 3

$b \cdot c^2 = \begin{pmatrix} 1/4 \\ 3/4 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 4/9 \end{pmatrix} = \frac{1}{3}$ (OK) $\quad Ac = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad b \cdot Ac = 0$ (NO)

ESEMPIO APPLICAZIONE

$$\begin{cases} y' = x^2 + e^y \\ y(0) = 1 \end{cases}$$

$$\begin{array}{c|cc} 0 & & \\ 2/3 & 2/3 & \\ \hline & 1/4 & 3/4 \end{array}$$

Metodo RK

$$y_{k+1} = y_k + \frac{1}{4} K_1 + \frac{3}{4} K_2$$

$$K_1 = h\, f\left(x_k + c_1 h,\ y_k + \sum_{j=1}^{2} a_{1j} K_j\right) = h\, f(x_k, y_k)$$

$$= \boxed{h\left(x_k^2 + e^{y_k}\right)}$$

$$K_2 = h\, f\left(x_k + c_2 h,\ y_k + \sum_{j=1}^{2} a_{2j} K_j\right) = h\, f\left(x_k + \frac{2}{3} h,\ y_k + \frac{2}{3} K_1\right)$$

$$= \boxed{\left(x_k + \frac{2}{3} h\right)^2 + e^{y_k + \frac{2}{3} K_1}}$$

Estensione al caso vettoriale

$$\begin{cases} \underline{Y}' = F(x, \underline{Y}) \\ \underline{Y}(0) = \underline{Y}_0 \end{cases} \qquad \underline{Y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \qquad F(x, \underline{Y}) = \begin{pmatrix} f_1(x, y_1, y_2, \dots, y_n) \\ f_2(x, y_1, y_2, \dots, y_n) \\ \vdots \\ f_n(x, y_1, y_2, \dots, y_n) \end{pmatrix}$$

Tutte le formule/metodi visti fino ad ora valgono nel caso vettoriale

Esempio (Eulero Esplicito)

$$y_{k+1} = y_k + h f(x_k, y_k)$$

$$y(x_{k+1}) - y(x_k) - h f(x_k, y(x_k)) = \tau_k$$

$$|\tau_k| \leq c h^2$$

$$\underline{Y}_{k+1} = \underline{Y}_k + h F(x_k, \underline{Y}_k)$$

$$\underline{Y}(x_{k+1}) - \underline{Y}(x_k) - h F(x_k, \underline{Y}(x_k)) = \vec{\tau}_k$$

$$\|\vec{\tau}_k\| \leq c h^2$$

Esempio ODE vettoriale

$$\begin{cases} x'(t) = y(t) \\ y'(t) = -x(t) \\ x(0) = 1 \\ y(0) = 0 \end{cases} \qquad z = \begin{pmatrix} x \\ y \end{pmatrix} \qquad F(t,z) = \begin{pmatrix} z_2 \\ -z_1 \end{pmatrix} = \begin{pmatrix} y \\ -x \end{pmatrix}$$

$$z(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

$$\Updownarrow$$

$$\begin{cases} z' = F(t,z) = \begin{pmatrix} z_2 \\ -z_1 \end{pmatrix} \\ z(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{cases}$$

$$g(t) = x(t)^2 + y(t)^2$$

$$g'(t) = 2x(t)x'(t) + 2y(t)y'(t)$$

$$= 2x(t)y(t) - 2y(t)x(t) = 0$$

$$g(t) = c \qquad \underbrace{x(t)^2 + y^2(t) = C}_{\text{cerchio}}$$

Eulero Esplicito

$$z_{k+1} = z_k + h F(t_k, z_k)$$

$$\begin{pmatrix} x_{k+1} \\ y_{k+1} \end{pmatrix} = \begin{pmatrix} x_k \\ y_k \end{pmatrix} + h \begin{pmatrix} y_k \\ -x_k \end{pmatrix}$$

$$\begin{cases} x_{k+1} = x_k + h y_k \\ y_{k+1} = y_k - h x_k \end{cases}$$

Eulero implicito

$$z_{k+1} = z_k + h\, F(t_{k+1}, z_{k+1})$$

$$\begin{pmatrix} x_{k+1} \\ y_{k+1} \end{pmatrix} = \begin{pmatrix} x_k \\ y_k \end{pmatrix} + h \begin{pmatrix} y_{k+1} \\ -x_{k+1} \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & -h \\ h & 1 \end{pmatrix} \begin{pmatrix} x_{k+1} \\ y_{k+1} \end{pmatrix} = \begin{pmatrix} x_k \\ y_k \end{pmatrix}$$

$\det = 1 + h^2 > 0$

Esercizio:

Scrivere passo di avanzamento per metodo RK di Heun e Collatz

Domanda: che succede a $g(t)$ integrale primo nella sua versione discreta?

$$g_k = x_k^2 + y_k^2$$

CASO EULERO ESPLICITO

$g_k = x_k^2 + y_k^2 \qquad g_{k+1}\ ?$

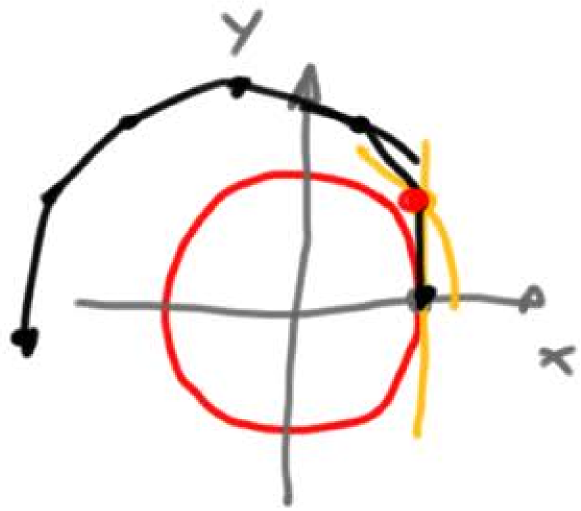
$$x_{k+1} = x_k + h\, y_k \qquad (x_k)$$
$$y_{k+1} = y_k - h\, x_k \qquad (y_k)$$

$$x_k x_{k+1} + y_k y_{k+1} = x_k^2 + y_k^2$$

$$\begin{pmatrix} x_k \\ y_k \end{pmatrix} \cdot \begin{pmatrix} x_{k+1} \\ y_{k+1} \end{pmatrix} = \left\| \begin{pmatrix} x_k \\ y_k \end{pmatrix} \right\|^2 \qquad |a \cdot b| \leq \|a\| \, \|b\|$$

$$\leq \left\| \begin{pmatrix} x_k \\ y_k \end{pmatrix} \right\| \left\| \begin{pmatrix} x_{k+1} \\ y_{k+1} \end{pmatrix} \right\|$$

$$\Rightarrow \left\| \begin{pmatrix} x_{k+1} \\ y_{k+1} \end{pmatrix} \right\| \geq \left\| \begin{pmatrix} x_k \\ y_k \end{pmatrix} \right\|$$



ESERCIZIO
con EULERO implicito $\left\| \begin{pmatrix} x_{k+1} \\ y_{k+1} \end{pmatrix} \right\| \leq \left\| \begin{pmatrix} x_k \\ y_k \end{pmatrix} \right\|$

STABILITA'

Si studia il comportamento di un metodo numerico sulla particolare ODE

$$\begin{cases} y' = \alpha y \\ y(0) = 1 \end{cases}$$

con α numero complesso.

PERCHE?

$z' = F(t,z)$ cerco andamento metodo numerico e soluzione esatta in un intorno di (t_0, z_0)

$z' = A(z-z_0) + b(t-t_0)$ $\Rightarrow$ LINEARIZZO

$w = z - z_0$

$w' = Aw + b(t-t_0)$

$$F(t,z) = F(t_0,z_0) + \overbrace{\boxed{\frac{\partial F}{\partial t_0}(t_0,z_0)}}^{b}(t-t_0) + \overbrace{\boxed{\frac{\partial F}{\partial z}(t_0,z_0)}}^{A}(z-z_0) + E$$

Se $b=0$ possiamo studiare una ODE del tipo

$$W' = AW$$

se A è matrice DIAGONALIZZABILE cioè esiste matrice T tale che

$$T^{-1}AT = \Lambda \qquad \Lambda = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}$$

(Gli argomenti seguenti si possono adattare nel caso A non sia diagonalizzabile usando la forma normale di Jordan)

$$T^{-1}W' = T^{-1}AW \qquad TV = W \qquad \text{(cambio di variabile)}$$

$$T^{-1}TV' = T^{-1}ATV$$

$$V' = \Lambda V \implies \begin{cases} v'_1 = \lambda_1 v_1 \\ v'_2 = \lambda_2 v_2 \\ \vdots \\ v'_n = \lambda_n v_n \end{cases}$$

n - ODE disaccoppiate

λ possono essere complessi

STABILITÀ PER EULERO ESPLICITO

$$\begin{cases} y' = \alpha y \\ y(0) = 1 \end{cases} \Rightarrow$$ Soluzione esatta $y(x) = e^{\alpha x}$ (anche se α complesso)

$y'(x) = \alpha e^{\alpha x} = \alpha y(x)$

nel caso α complesso

$$e^{\alpha x} = e^{Re(\alpha)x} \left(\cos Im(\alpha) x + i \sin Im(\alpha) x \right)$$

$|e^{\alpha x}| \to 0$ se $Re(\alpha) < 0$ $\quad$ $Re(\alpha) = 0$ $\quad |e^{\alpha x}| = 1$

$Re(\alpha) > 0 \quad |e^{\alpha x}| \to \infty$

$$y_{n+1} = y_n + h \alpha y_n = (1 + \alpha h) y_n$$

Domanda $y_n \to ?$ per $n \to \infty$

Vorremmo che $y_n \to 0$ se $Re(\alpha) < 0$

Succede sempre?