STABILITÀ

$$\begin{cases} y' = \alpha y \\ y(0) = 1 \end{cases} \qquad Re(\alpha) < 0$$

α numero complesso

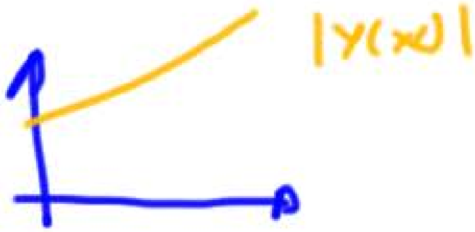
soluzione esatta

$$y(x) = e^{\alpha x} \qquad \alpha = a + ib$$

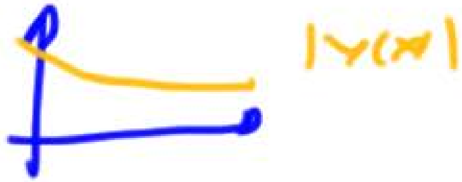
$$y(x) = e^{ax + ibx} = e^{ax}(\cos bx + i \operatorname{sen} bx)$$

$$|y(x)| = e^{ax}\,|\cos bx + i \operatorname{sen} bx| = e^{ax}\sqrt{(\cos bx)^2 + (\operatorname{sen} bx)^2} = e^{ax}$$

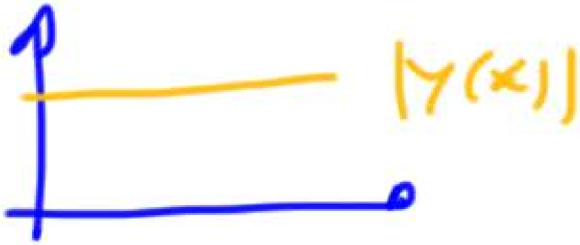
se $a > 0$



$a < 0$



se $a = 0$



Eulero Esplicito

$$\begin{cases} y' = \alpha y \\ y(0) = 1 \end{cases} \qquad \begin{cases} y_{k+1} = y_k + h\alpha y_k = y_k(1+h\alpha) \\ y_0 = 1 \end{cases}$$

$$y_0 = 1$$
$$y_1 = y_0(1+h\alpha) = 1+\alpha h$$
$$y_2 = y_1(1+h\alpha) = (1+\alpha h)^2$$
$$\vdots$$
$$y_k = (1+\alpha h)^k$$

$$|y_k| = |1+\alpha h|^k = |1+(a+ib)h|^k = |1+ah+ibh|^k$$

$$= \Big(\underbrace{\sqrt{(1+ah)^2+(bh)^2}}_{A}\Big)^k = A^k$$

se $A>1$ $\quad A^k \to \infty$, $k \to \infty$

se $A<1$ $\quad A^k \to 0$, $k \to \infty$

$|y_k| = A^k \qquad A = \sqrt{(1+ah)^2+(bh)^2}$

se $A>1 \quad \displaystyle\lim_{k\to\infty} A^k \to \infty$

$A<1 \quad \displaystyle\lim_{k\to\infty} A^k \to 0$

Metodo Numerico IDEALE

se $Re(\alpha)<0$ allora $A<1$ (decadimento)

$Re(\alpha)=0$ allora $A=1$

$Re(\alpha)>0$ allora $A>1$ (crescita esp.)

Questo non succede in generale per un metodo numerico ma può succedere che $Re(\alpha)<0$ e la soluzione esplode o viceversa $Re(\alpha)>0$ e la soluzione numerica decade

Esempio

$$\begin{cases} y' = -y \\ y(0) = 1 \end{cases} \qquad y(x) = e^{-x}$$

Metodo Eulero Esplicito

$$y_{k+1} = y_k - h y_k = (1-h) y_k$$

$$y_0 = 1$$
$$y_1 = 1-h$$
$$y_2 = (1-h)^2$$

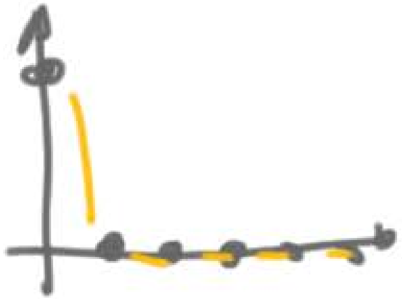
$$y_k = (1-h)^k$$

$0 < h < 1 \qquad 0 < 1-h < 1$

$$y_k = (1-h)^k$$

$$\begin{cases} y_k \to 0 \\ \quad k \to \infty \\ y_k > 0 \end{cases}$$

$h = 1 \qquad 1-h = 0$



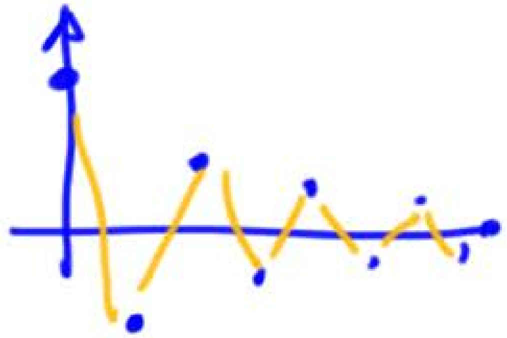
$$\begin{cases} y_0 = 1 \\ y_k = 0 \quad k = 1, 2 \ldots \end{cases}$$

$$\begin{cases} y_k \to 0 \\ \quad k \to \infty \\ y_k \geq 0 \end{cases}$$

non più strettamente positivo

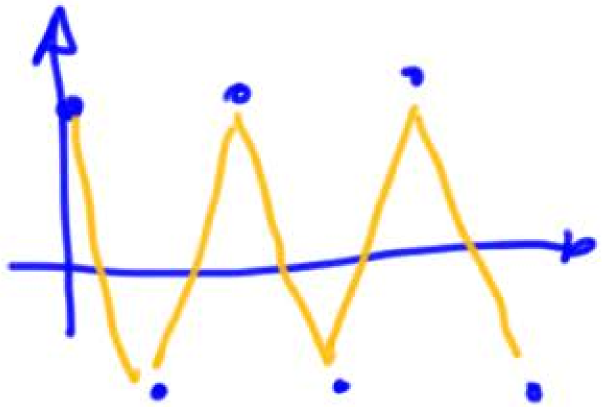
$1 < h < 2$ $\qquad$ $0 < h-1 < 1$

$y_k = (1-h)^k$ $\qquad$ $(1-h) \in (-1,0)$



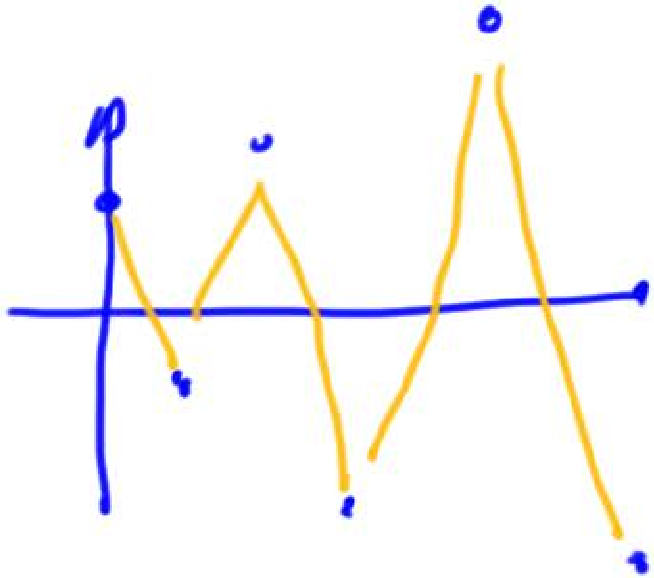
$$\begin{cases} y_k \to 0 \\ k \to \infty \\ y_k \text{ sia positivo da negativo} \end{cases}$$

$h = 2$ $\qquad$ $1-h = -1$



$y_k = (-1)^k$

$h > 2$ $\qquad$ $|1-h| > 1$



y_k oscilla $\pm$ e cresce esponenzialmente

Eulero implicito

$$\begin{cases} y' = -y \\ y(0) = 1 \end{cases}$$

$$y_{k+1} = y_k - h\, y_{k+1}$$

$$(1+h)\, y_{k+1} = y_k$$

$$y_{k+1} = \frac{y_k}{1+h}$$

per $h > 0$

$$0 < \frac{1}{1+h} < 1$$

$y_0 = 1$

$y_1 = \frac{1}{1+h}$

$y_2 = \frac{1}{(1+h)^2}$

$\vdots$

$y_k = \frac{1}{(1+h)^k} = \left(\frac{1}{1+h}\right)^k$

$y_k \to 0$

$k \to \infty$

$y_k > 0$

CRANK-NICOLSON

$$\begin{cases} y' = f(x,y) \\ y(a) = y_0 \end{cases} \qquad y_{k+1} = y_k + \frac{h}{2}\left[f(x_k, y_k) + f(x_{k+1}, y_{k+1}) \right]$$

$$\begin{cases} y' = -y \\ y(0) = 1 \end{cases} \qquad y_{k+1} = y_k + \frac{h}{2}\left[-y_k - y_{k+1} \right]$$

$$y_{k+1}\left(1 + \frac{h}{2}\right) = y_k\left(1 - \frac{h}{2}\right)$$

$$y_{k+1} = \frac{2-h}{2+h}\, y_k$$

$y_0 = 1$

$y_1 = \frac{2-h}{2+h}$

$y_2 = \left(\frac{2-h}{2+h}\right)^2$

$y_k = \left(\frac{2-h}{2+h}\right)^k$

Se $0 < h < 2$

$$0 < \frac{2-h}{2+h} < 1$$

$y_k \to 0$

$k \to \infty$

$y_k > 0$

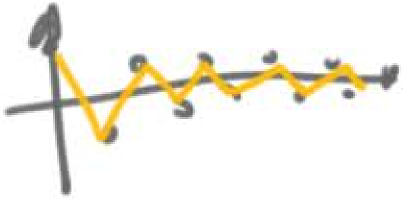
Se $h = 2$

$y_k = 0 \quad k \geq 1$

Se $h > 2$

$|2-h| < |2+h|$

$$-1 < \frac{2-h}{2+h} < 0$$



Eulero Esplicito

$$\begin{cases} y' = \alpha y \\ y(0) = 1 \end{cases} \qquad \begin{cases} y_{k+1} = y_k + h\alpha y_k \\ y_0 = 1 \end{cases} \qquad \alpha = a + ib$$

$$|y_k| = A^k$$

$$A = \sqrt{(1+ah)^2 + (bh)^2}$$

Se $\alpha < 0$ $\quad |y(x)| \to 0$, $x \to \infty$

vorremmo che $A < 1$

cerchiamo h tale che $A^2 = 1$

$$\cancel{1} = A^2 = (1+ah)^2 + b^2h^2 = 1 + a^2h^2 + 2ah + b^2h^2$$

$$= h^2(a^2+b^2) + 2ah \cancel{+1}$$

$$h\left(|\alpha|^2 h + 2\,Re(\alpha)\right) = 0 \Rightarrow h = -\frac{2\,Re(\alpha)}{|\alpha|^2}$$

$|A| < 1$ per $h > 0$ $\quad h < -\frac{2\,Re(\alpha)}{|\alpha|^2} = \frac{-2a}{a^2+b^2}$

Eulero Esplicito

$$\begin{cases} y' = \alpha y \\ y(0) = 1 \end{cases} \qquad \begin{cases} y_{k+1} = y_k + \underbrace{(h\alpha)}_{s} y_k = y_k + s\, y_k \\ y_0 = 1 \end{cases}$$

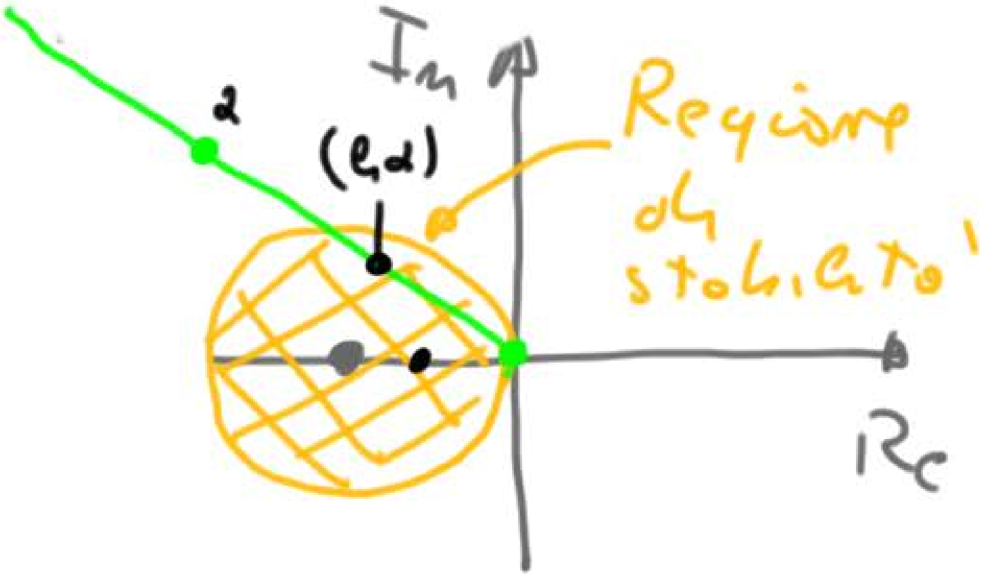
$$y_{k+1} = (1+s)\, y_k \qquad y_k = (1+s)^k\, y_0$$

$$|y_k| \le 1 \qquad |1+s| \le 1$$

$$s = a + ib \qquad |1 + a + ib|^2 \le 1^2$$

$$(1+a)^2 + b^2 \le 1$$

$$(1+a)^2 + b^2 = 1$$

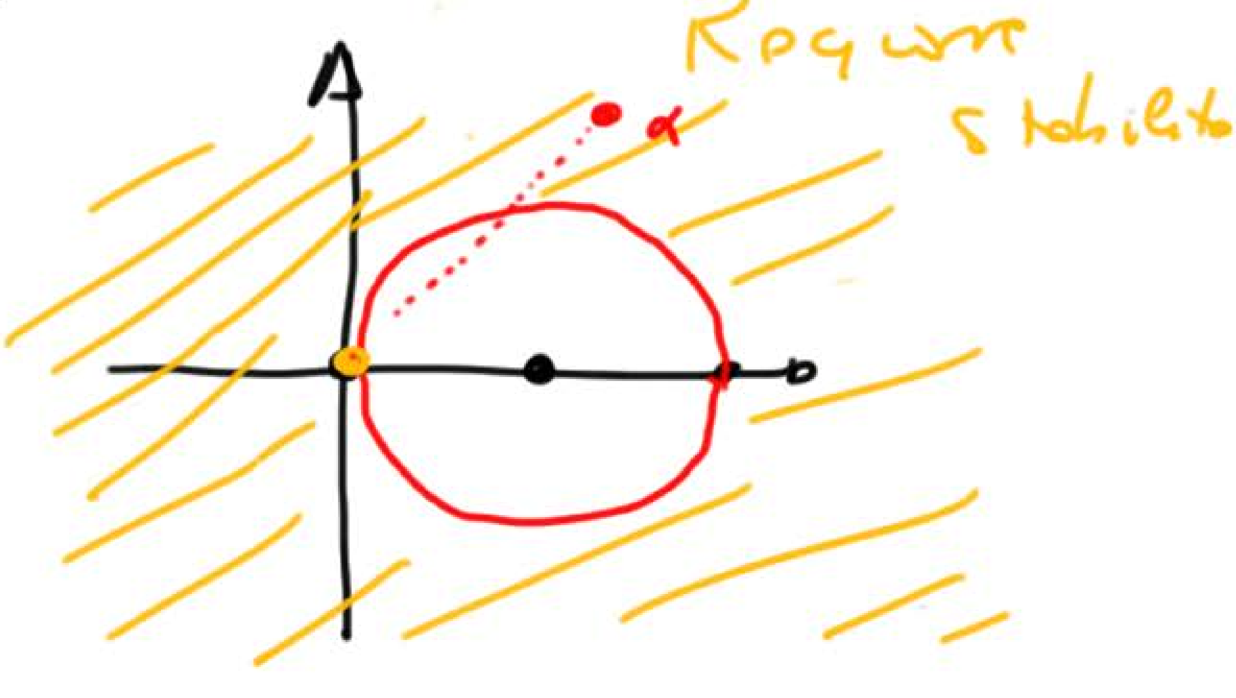


Eulero Implicito

$$\begin{cases} y' = \alpha y \\ y(0) = 1 \end{cases} \qquad \begin{cases} y_{k+1} = y_k + \underbrace{(h\alpha)}_{s} y_{k+1} \\ y_0 = 1 \end{cases} \qquad y_{k+1}(1-s) = y_k$$

$$|y_n| \le 1 \qquad y_n = \frac{y_0}{(1-s)^k} \qquad |1-s| \ge 1$$

$$s = a + ib \qquad |1 - a - ib|^2 = (1-a)^2 + b^2 \ge 1$$



REGIONE STABILITA' C.N.

$$\begin{cases} y' = \alpha y \\ y(0) = 1 \end{cases} \qquad y_{n+1} = y_n + \frac{h\alpha}{2}(y_n + y_{n+1}) \qquad s = h\alpha$$

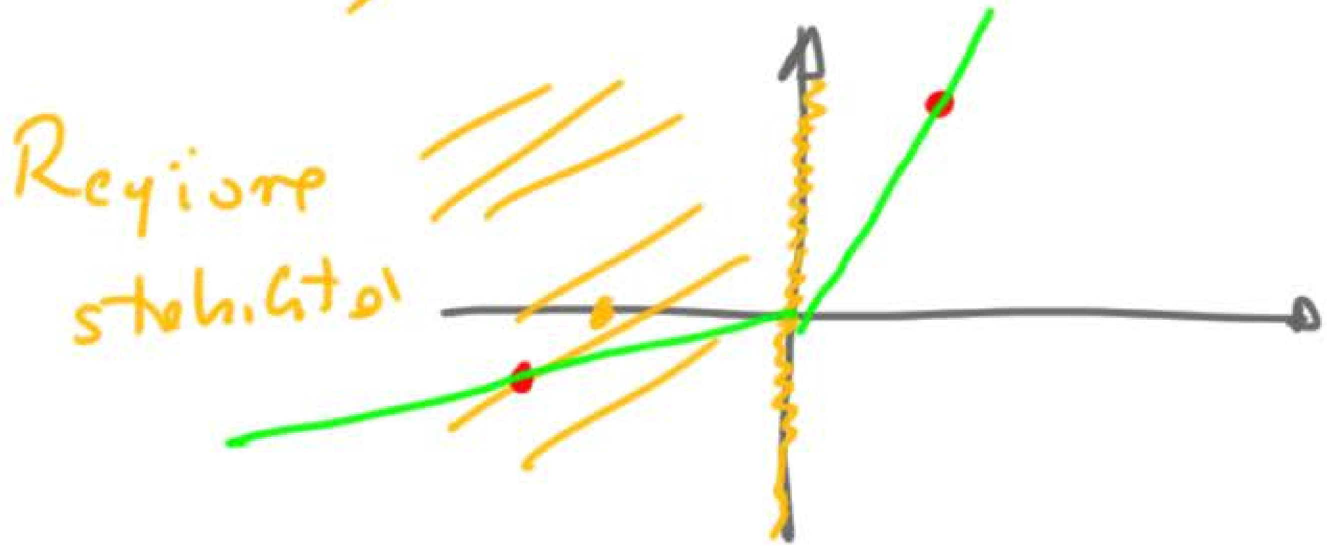
$$y_{n+1} = \left(\frac{2+s}{2-s}\right) y_n$$

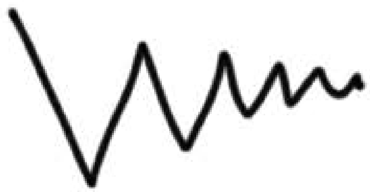
FATTORE AMPLIFICAZIONE

$$\left|\frac{2+s}{2-s}\right| \leq 1 \qquad |2+s| \leq |2-s| \qquad \text{bordo} \quad |2+s|^2 = |2-s|^2$$

$$s = a + ib \qquad (2+a)^2 + \cancel{b^2} = (2-a)^2 + \cancel{b^2}$$

$$\cancel{4} + \cancel{a^2} + 4a = \cancel{4} + \cancel{a^2} - 4a \quad \Rightarrow \quad a = 0$$





REGIONE STABILITÀ METODO
NON FUNZIONANTE