# METODI MULTISTEP

$$\begin{cases} y' = f(x,y) \\ y(0) = y_0 \end{cases}$$

$y_n \simeq y(x_n)$

$x_k = a + hk$

$h = \frac{b-a}{m}$

$$y_{k+1} = \phi(x_k, \underbrace{y_k, y_{k-1}, \ldots, y_{k-e}}_{\text{passi precedenti}}, h)$$

GENERICO MULTISTEP LINEARE

$$\sum_{j=-1}^{e} \alpha_j y_{k-j} = h \sum_{j=-1}^{e} \beta_j f(x_{k-j}, y_{k-j})$$

$e = 0$ $\quad \alpha_{-1} = 1 \quad \alpha_0 = 0$ $\quad y_{k+1} - y_k = h f(x_k, y_k)$

$\beta_{-1} = 0 \quad \beta_0 = 1$ $\quad$ Eulero esplicito

# GENERICO MULTISTEP LINEARE

$$\sum_{j=-1}^{e} \alpha_j y_{n-j} = h \sum_{j=-1}^{e} \beta_j f(x_{n-j}, y_{n-j})$$

$$\underbrace{\alpha_{-1} y_{n+1} - h \beta_{-1} f(x_{n+1}}, y_{n+1}) = \underbrace{\sum_{j=0}^{e} (\alpha_j y_{n-j} + h \beta_j f(x_{n-j}, y_{n-j}))}_{r}$$

$$f(y) = \alpha_{-1} y - h \beta_{-1} f(x_{n+1}, y)$$

devo risolvere $f(y) = r \Rightarrow y_{n+1}$

se $\beta_{-1} = 0 \Rightarrow$ soluzione immediata

$\alpha_{-1} \neq 0$

# Esempi di metodi multistep

$$\begin{array}{ccc} \alpha = ( & 1 & -1 \,) \\ & \alpha_{-1} & \alpha_0 \end{array} \qquad \begin{array}{ccc} \beta = ( & 1 & 0 \,) \\ & \beta_{-1} & \beta_0 \end{array}$$

$$y_{k+1} - y_k = h\, f(x_{k+1}\, y_{k+1})$$

(Eulero implicito)

$$\begin{array}{cccc} \alpha = ( & 1 & -\frac{4}{3} & \frac{1}{3} \,) \\ & -\alpha_{-1} & \alpha_0 & \alpha_1 \end{array} \qquad \begin{array}{cccc} \beta = ( & 1 & 0 & 0 \,) \\ & \beta_{-1} & \beta_0 & \beta_1 \end{array}$$

$$y_{k+1} - \frac{4}{3} y_k + \frac{1}{3} y_{k-1} = h\, f(x_{k+1}, y_{k+1})$$

$$y - h\, f(x_{k+1}, y) = \frac{4}{3} y_k - \frac{1}{3} y_{k-1}$$

$y_{k+1} = y$ soluzione problema

$$\begin{array}{ccccc} \alpha = ( & 1 & -\frac{18}{11} & \frac{9}{11} & -\frac{2}{11} \,) \\ & \alpha_{-1} & \alpha_0 & \alpha_1 & \alpha_2 \end{array} \qquad \begin{array}{ccccc} \beta = ( & 1 & 0 & 0 & 0 \,) \\ & \beta_{-1} & \beta_0 & \beta_1 & \beta_2 \end{array}$$

$$y_{k+1} - \frac{18}{11} y_k + \frac{9}{11} y_{k-1} - \frac{2}{11} y_{k-2} = h\, f(x_{k+1}, y_{k+1})$$

RICORDI DA METODI A SINGOLO PASSO

$$y_{k+1} = \phi(x_k, y_k, h)$$

$$y(x_{k+1}) - \phi(x_k, y(x_k), h) = \tau_h \Rightarrow \text{ERRORE LOCALE}$$

$$\tau_h = O(h^{p+1})$$

in modo che $|y(x_k) - y_k| = O(h^p)$

CALCOLO ERRORE LOCALE TRONCAMENTO METODI MULTISTEP

$$\sum_{j=-1}^{\ell} \alpha_j y(x_{k-j}) - h \sum_{j=-1}^{\ell} \beta_j f(x_{k-j}, y(x_{k-j})) = \tau_h$$

CALCOLO ERRORE LOCALE TRONCAMENTO METODI MULTISTEP

$$\sum_{j=-1}^{\ell} \alpha_j y(x_{k-j}) - h \sum_{j=-1}^{\ell} \beta_j f(x_{k-j}, y(x_{k-j})) = \tau_h$$

$$x_{k-j} = a + h(k-j) = \underline{a + hk} - jh = x_k - jh$$

$$y(x_{k-j}) = y(x_k) - jh\, y'(x_k) + \frac{(-jh)^2}{2} y''(x_k) + \frac{(-jh)^3}{3!} y'''(x_k) + \dots + \frac{(-jh)^p}{p!} y^{(p)}(x_k) + O(h^{p+1})$$

$$f(x_{k-j}, y(x_{k-j})) = y'(x_{k-j})$$

$$y'(x_{k-j}) = y'(x_k) - jh\, y''(x_k) + \frac{(-jh)^2}{2} y'''(x_k) + \dots + \frac{(-jh)^{p-1}}{(p-1)!} y^{p}(x_k) + O(h^p)$$

$$\sum_{j=-1}^{\ell} \alpha_j y(x_{k-j}) - h \sum_{j=-1}^{\ell} \beta_j f(x_{k-j}, y(x_{k-j})) = \tau_h$$

$$y(x_{n-j}) = y(x_n) - hj\, y'(x_n) + O(h^2)$$

$$f(x_{k-j}, y(x_{k-j})) = y'(x_{k-j}) = y'(x_n) + O(h)$$

$$\sum_{j=-1}^{\ell} \alpha_j \left( y(x_n) - hj\, y'(x_n) + O(h^2) \right)$$

$$- h \sum_{j=-1}^{\ell} \beta_j \left( y'(x_k) + O(h) \right) = \tau_h$$

$$y(x_k) \underbrace{\left( \sum_{j=-1}^{\ell} \alpha_j \right)}_{=0} + y'(x_k) \underbrace{\sum_{j=-1}^{\ell} \left( (-hj)\alpha_j - h\beta_j \right)}_{=0} + O(h^2) = \tau_h$$

$$\sum_{j=-1}^{\ell} \alpha_j = 0 \qquad \sum_{j=-1}^{\ell} (j\alpha_j + \beta_j) = 0$$

# In generale

Sviluppo Taylor per $y(x_{k-j}) = y(x_k) + \dots + O(h^{p+1})$

$f(x_{k-j}, y(x_{k-j})) = y'(x_{k-j})$

$y'(x_{k-j}) = y'(x_k) + \quad + O(h^p)$

$\Rightarrow$ sostituire in $\sum_{j=-1}^{\ell} \alpha_j y(x_{k-j}) - h \sum_{j=-1}^{\ell} \dots = \tau_h$

$\Rightarrow$ raccogliere

$$\left(\sum_{j=-1}^{\ell} \alpha_j\right)$$

$$y(x_k)\,[\,(0)\,] +$$
$$h\, y'(x_k)\,[\,(1)\,] +$$
$$h^2 y''(x_k)\,[\,(2)\,] +$$
$$h^p y^{(p)}(x_k)\,[\,(p)\,] + O(h^{p+1}) = \tau_h$$

# USO DEL BARBATRUCCO

$y' = x^{m-1}$ $\qquad$ $m = 1$ $\quad y' = 1$ $\qquad$ $m = 0$ $\quad y' = 0$

$\Downarrow$ $\qquad\qquad\qquad\qquad$ $y(x) =$ costante $= 1$

$$y(x) = \frac{x^m}{m}$$

se il metodo ha ordine $p$

$$\tau_h = y^{(p+1)}(\eta)\, h^{p+1}\, C$$

se $y(x)$ soluzione esatta è polinomio

$y(x) = a_0 + a_1 x + \quad + a_p x^p$ $\qquad$ $y^{(p+1)}(x) = 0$

e quindi $\tau_h = 0$

- scelgo soluzione $y(x) = 1$ $\quad \tau_h = 0 \Rightarrow \tau_h = O(h)$
- scelgo soluzione $y(x) = x$ $\quad \tau_h = 0 \Rightarrow \tau_h = O(h^2)$
- " " $y(x) = x^2$ $\quad \tau_h = 0 \Rightarrow \tau_h = O(h^3)$

Caso $y(x) = 1 \quad y'(x) = 0 \quad (k=0)$

$$\underbrace{\sum_{j=-1}^{\ell} \alpha_j y(x-j)}_{\displaystyle\sum_{j=-1}^{\ell} \alpha_j = 0} - h \cancel{\sum_{j=-1}^{\ell} \beta_j y'(x-j)} = 0$$

Caso $y(x) = x \quad y'(x) = 1 \quad (k=0)$

$$\sum_{j=-1}^{\ell} \alpha_j y(x-j) - h \sum_{j=-1}^{\ell} \beta_j y'(x-j) = 0$$

$$\sum_{j=-1}^{\ell} \alpha_j (-j) - h \sum_{j=-1}^{\ell} \beta_j = 0$$

$$-\sum_{j=-1}^{\ell} (j\alpha_j + \beta_j) = 0$$

Caso $y(x) = x^m$ $\quad y'(x) = m x^{m-1}$ $\quad k = 0$

$$\sum_{j=-1}^{e} \alpha_j y(x_{-j}) - h \sum_{j=-1}^{r} \beta_j y'(x_{-j}) = 0$$

$$\sum_{j=-1}^{e} \alpha_j (-jh)^m - h \sum_{j=-1}^{e} \beta_j m (-jh)^{m-1} = 0$$

$$\sum_{j=-1}^{e} \left( \alpha_j (-j)^m - \beta_j m (-j)^{m-1} \right) = 0$$

$$\sum_{j=-1}^{e} \left( \alpha_j (-1)^m j^m - \beta_j m (-1)^{m-1} j^{m-1} \right) = 0$$

$$\sum_{j=-1}^{e} (-1)^m \left( \alpha_j j^m + \beta_j m j^{m-1} \right) = 0$$

$$\Rightarrow \sum_{j=-1}^{e} \left( \alpha_j j^m + m \beta_j j^{m-1} \right) = 0$$

## Come si usano

$\alpha = (\alpha_{-1}\ \alpha_0\ \ldots\ \alpha_e)$ $\quad \beta = (\beta_{-1}\ \beta_0\ \ldots\ \beta_e)$

$$\sum_{j=-1}^{e} \alpha_j = 0$$

deve essere verificato altrimenti metodo NON consistente cioè $\tau_h$ è meno di $O(h)$

$$\sum_{j=-1}^{e} (j\alpha_j + \beta_j) = 0$$

deve essere verificato se sì $\Rightarrow$ $p = 1$ cioè $\tau_h = O(h^2)$

$$\sum_{j=-1}^{e} (j^2\alpha_j + 2j\beta_j) = 0$$

se non verificato $\Rightarrow$ stop ordine 1

se verificato $\tau_h$ almeno $O(h^3)$

$\vdots$

$$\sum_{j=-1}^{e} (j^r\alpha_j + r j^{r-1}\beta_j) = 0$$

Se verificato per $r = 0, 1, 2, \ldots, p$ ma non per $r = p+1$ ordine $p$ $\quad \tau_h = O(h^{p+1})$

# Esempio

$\alpha = (1 \quad -1) \qquad \beta = (0 \quad 1)$ Eulero esplicito

$$\sum_{j=-1}^{0} \alpha_j = 1 + (-1) = 0 \qquad (\text{OK})$$

$$\sum_{j=-1}^{0} (j\alpha_j + \beta_j) = (-1)\,1 + 0\,(-1) + 1 = 0 \qquad (\text{OK})$$

$$\sum_{j=-1}^{0} (j^2\alpha_j + 2j\beta_j) = 1\,1 + 0\,(-1) + 2(-1)\,0 + 2\,0\,1 = 1$$

(<u>NO</u>)