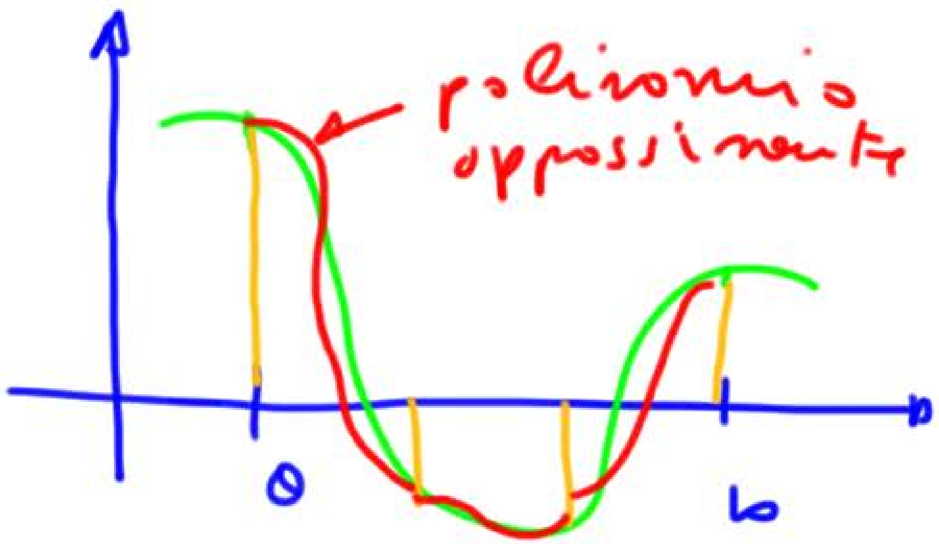
FORMULE QUADRATURA



$$\int_a^b f(x)\,dx = \sum_{k=1}^{N} \int_{x_{k-1}}^{x_k} f(x)\,dx$$

$$H = \frac{b-a}{N} \qquad = \sum_{k=1}^{N} \int_{x_{k-1}}^{x_k} P_k(x)\,dx + \text{Errore}$$

Esempio

$$P_k(x) = f[x_{k-1}] + f[x_{k-1}, x_k](x - x_{k-1}) \qquad \text{(Retta)}$$

$$\int_{x_{k-1}}^{x_k} P_k(x)\,dx = f(x_{k-1})\overbrace{(x_k - x_{k-1})}^{H} + f[x_{k-1}, x_k]\frac{\overbrace{(x_k - x_{k-1})^2}^{H^2}}{2} = \frac{f(x_{k-1}) + f(x_k)}{2}H$$

Esempio

$$P_k(x) = f[x_{k-1}] + f[x_{k-1}, x_k](x - x_{k-1}) + f\left[x_{k-1}, x_k, \frac{x_k + x_{k-1}}{2}\right](x - x_{k-1})(x - x_k)$$

$= x_{k-\frac{1}{2}}$

$$\int_{x_{k-1}}^{x_k} P_k(x) = \frac{f(x_{k-1}) + 4 f(x_{k-\frac{1}{2}}) + f(x_k)}{6} H$$

Può essere conveniente definire $n = 2N$ intervallini di ampiezza $h = \frac{H}{2} = \frac{b-a}{n}$ ed

$\tilde{x}_k = a + hk$ in modo che

$$\int_{x_{k-1}}^{x_k} P_k(x)\,dx = \int_{\tilde{x}_{2k-2}}^{\tilde{x}_{2k}} P_k(x)\,dx$$

$P_k(x)$ interpola $f(x_{2k-2})$ $f(x_{2k-1})$ $f(x_{2k})$

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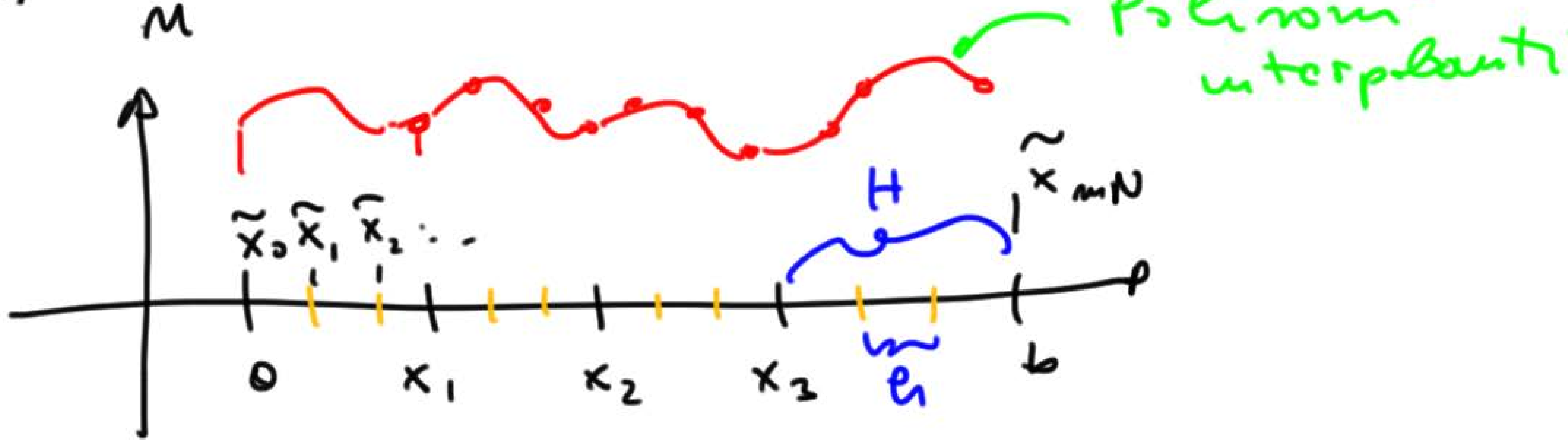
$$\int_{\tilde{x}_{2k-2}}^{\tilde{x}_{2k}} P_k(x)\,dx = \underbrace{\frac{h}{3}\left(f(\tilde{x}_{2k-2}) + 4\,f(\tilde{x}_{2k-1}) + f(\tilde{x}_{2k})\right)}_{\text{formula integrale approssimata con un pezzo di parabola}}$$

In generale

$$\int_a^b f(x)\,dx$$ viene approssimato con:

$$H = \frac{b-a}{N} \qquad x_k = a + kH \qquad k = 0 \dots N$$

$$h = \frac{b-a}{M} \qquad M = mN \qquad \tilde{x}_k = a + kh \qquad k = 0 \dots M$$



Integrale esatto

$\Downarrow$

Approssimato con la somma degli integrali esatti dei polinomi interpolanti

Problema: stimare errori

Per un singolo intervallo l'errore vale

$$\int_{x_{k-1}}^{x_k} f(x)\,dx = \int_{x_{k-1}}^{x_k} p_k(x)\,dx + \text{Errore}$$

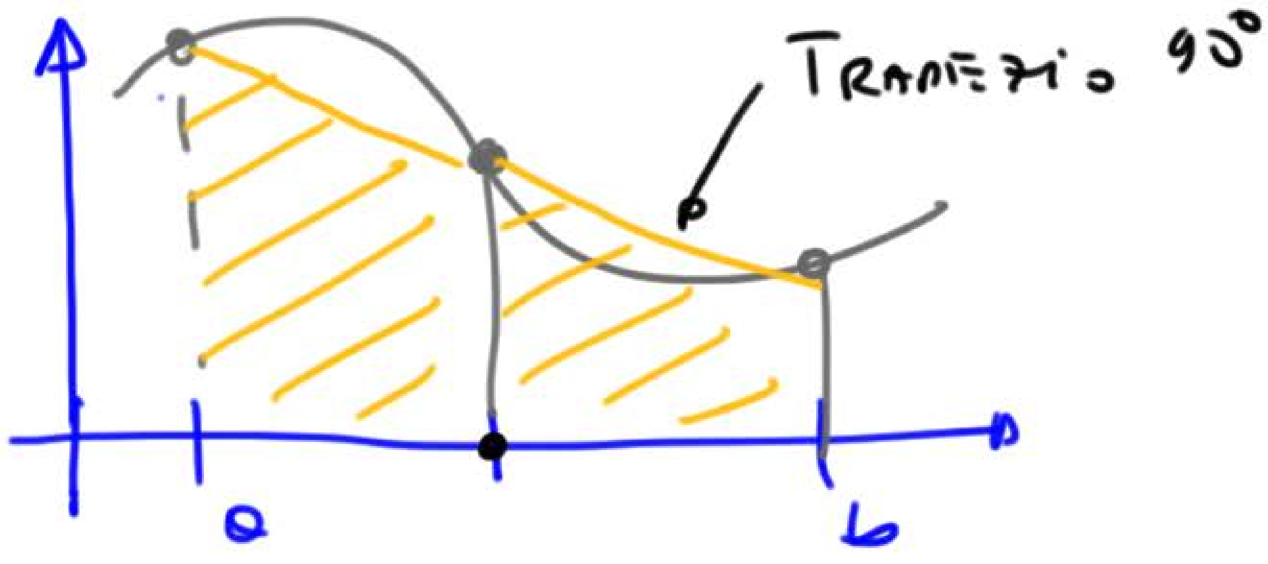
$$\text{Errore} =$$

$-\dfrac{h^3}{12} f''(\xi_k)$ nel caso di interpolazione lineare (METODO DEI TRAPEZI)

$-\dfrac{h^5}{90} f^{IV}(\xi_k)$ nel caso di interpolazione parabolica (METODO DI SIMPSON)

STIMA ERRORE DA FARE DOPO

Metodo dei trapezi



$$H = h$$

$$x_k = a + kh$$

$$\xi_k \in (x_{k-1}, x_k)$$

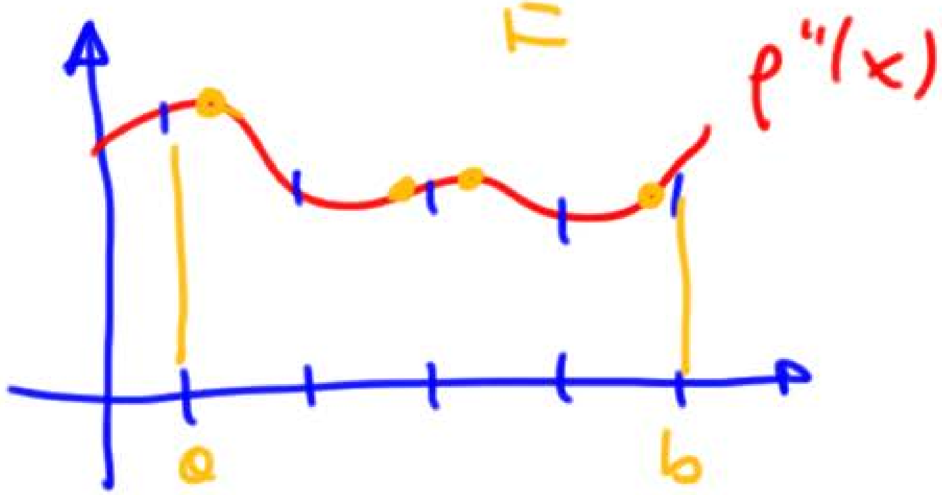
$$\int_a^b f(x)\,dx = \sum_{k=1}^{M} \left(\frac{f(x_{k-1}) + f(x_k)}{2} h - \frac{h^3}{12} f''(\xi_k) \right)$$

$$= \frac{f(a) + f(b)}{2} h + h \sum_{k=1}^{M-1} f(x_k)$$

$$- \frac{h^3}{12} \sum_{k=1}^{M} f''(\xi_k)$$

$$\int_a^b f(x)\,dx = \frac{f(a)+f(b)}{2}h + h\sum_{k=1}^{n-1} f(x_k) \underbrace{- \frac{h^3}{12}\sum_{k=1}^{n} f''(\xi_k)}_{E}$$

$$E = -\frac{nh^3}{12}\left(\frac{1}{n}\sum_{k=1}^{n} f''(\xi_k)\right)$$



$$\min a_k \le \left(\frac{1}{n}\sum_{k=1}^{n} a_k\right) \le \max a_k$$

$\Rightarrow \exists\, \xi$ tale che $\frac{1}{n}\sum_{k=1}^{n} f''(\xi_k) = f''(\xi) \quad \xi \in (a,b)$

$$hn = (b-a)$$

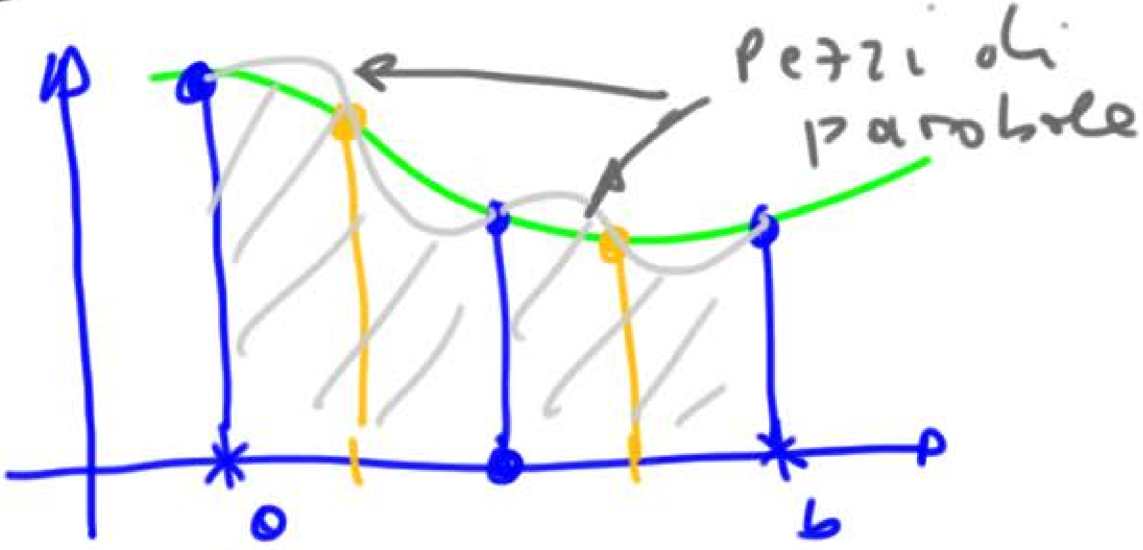
$$E = -\frac{h^2(b-a)}{12} f''(\xi)$$

FORMULA TRAPEZI

$$\int_a^b f(x)\,dx = \frac{f(a)+f(b)}{2}\,h + h\sum_{k=1}^{n-1} f(x_k)$$

$$-\ \frac{h(b-a)}{12}\,f''(\xi) \qquad \xi \in (a,b)$$

METODO DI SIMPSON



$H = 2h$

$x_k = a + kH$

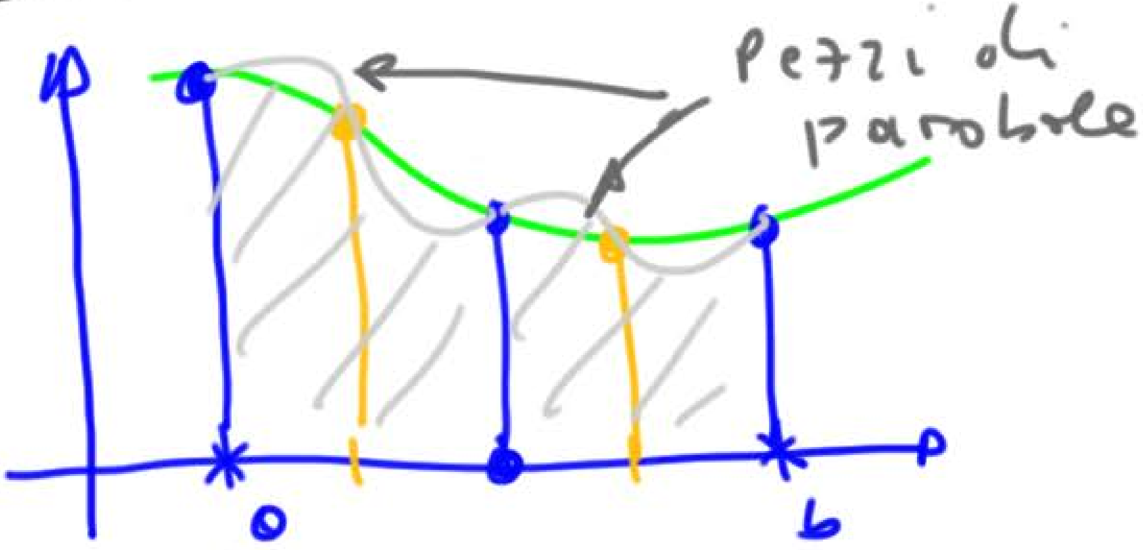
$\tilde{x}_k = a + kh$

$$\int_a^b f(x)\,dx = \sum_{k=1}^{N}\left(\frac{f(\tilde{x}_{2k-2}) + 4f(\tilde{x}_{2k-1}) + f(\tilde{x}_{2k})}{3}h - \frac{h^5}{90}f^{IV}(\xi_k)\right)$$

1 4 2 4 2 ... 4 1

$$= \frac{h}{3}\left(f(\tilde{x}_0) + 4f(\tilde{x}_1) + 2f(\tilde{x}_2) + 4f(\tilde{x}_3) + \dots + 4f(\tilde{x}_{M-1}) + f(\tilde{x}_M)\right) - \frac{h^5}{90}\sum_{k=1}^{N} f^{IV}(\xi_k)$$

METODO DI SIMPSON



$H = 2h$

$x_k = a + kH$

$\tilde{x}_k = a + kh$

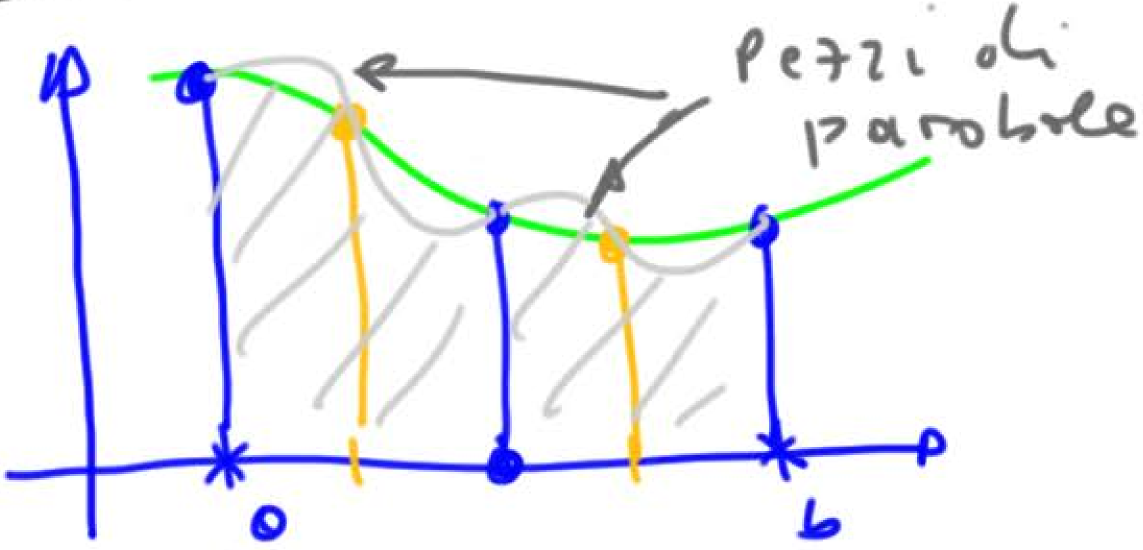
$$= \frac{h}{3}\left(f(\tilde{x}_0) + 4f(\tilde{x}_1) + 2f(\tilde{x}_2) + 4f(\tilde{x}_3) + \dots + 4f(\tilde{x}_{n-1}) + f(\tilde{x}_n)\right)$$

$$- \frac{h^5}{90}\sum_{k=1}^{N} f^{IV}(\xi_k)$$

$$- \frac{h^5}{90}\sum_{k=1}^{N} f^{IV}(\xi_k) = -\frac{Nh^5}{90}\overbrace{\left(\frac{1}{N}\sum_{k=1}^{N} f^{IV}(\xi_k)\right)}^{f^{IV}(\xi)} = -\frac{(b-a)h^4}{180} f^{IV}(\xi)$$

$$Nh = N\frac{b-a}{n} = N\frac{b-a}{2N} = \frac{1}{2}(b-a)$$

METODO DI SIMPSON



$H = 2h$

$x_k = a + kH$

$\tilde{x}_k = a + kh$

$$\int_a^b f(x)\,dx = \frac{h}{3}\left(f(\tilde{x}_0) + 4f(\tilde{x}_1) + 2f(\tilde{x}_2) + \quad + f(\tilde{x}_n)\right)$$

$$- \frac{h^4}{180}(b-a)\,f^{IV}(\xi)$$

$$= \sum_{k=0}^{m} w_k f(\tilde{x}_k) + \text{Errore}$$

$$w = \frac{h}{3}\left(1\ 4\ 2\ 4\ 2 \ldots\ 4\ 2\ 4\ 1\right)$$

ESEMPIO "TIPO ESAME SCRITTO"

Domanda: dato

$$\underline{I} = \int_1^{10} \cos x \, e^{-x} \, dx$$

quanti intervalli servono per calcolare I con errore E $|E| < 10^{-6}$ con TRAPEZI e SIMPSON.

$$E = -\frac{h^2}{12}(b-a) f''(\xi) \quad \text{per metodo dei TRAPEZI}$$

$$|E| \leq 10^{-6} \qquad \frac{h^2}{12}(b-a)\,|f''(\xi)| \leq 10^{-6}$$

Se trovo M tale che $|f''(z)| \leq M \quad z \in [a, b]$

$$|E| = \frac{h^2}{12}(b-a)\,|f''(\xi)| \leq \frac{h^2}{12}(b-a) M \leq 10^{-6}$$

$$\Rightarrow |E| \leq 10^{-6}$$

ESEMPIO "TIPO ESAME SCRITTO"

Domanda: dato

$$I = \int_1^{10} \cos x \, e^{-x} \, dx$$

quanti intervalli servono per calcolare I con errore E

$|E| < 10^{-6}$ con TRAPEZI e SIMPSON.

$$|E| \leq \frac{h^2}{12}(b-a) M \leq \varepsilon = 10^{-6} \qquad h = \frac{b-a}{n}$$

$$\frac{(b-a)^3}{12 n^2} M \leq \varepsilon \qquad n^2 \geq \frac{(b-a)^3 M}{12 \varepsilon}$$

$$n \geq \sqrt{\frac{(b-a)^3 M}{12 \varepsilon}}$$

$$|E| \leq \frac{h^2}{12}(b-a)\, M \leq \varepsilon = 10^{-6} \qquad h = \frac{b-a}{n}$$

$$\frac{(b-a)^3}{12\, n^2} M \leq \varepsilon \qquad n^2 \geq \frac{(b-a)^3 M}{12\, \varepsilon}$$

$$n \geq \sqrt{\frac{(b-a)^3 M}{12\, \varepsilon}}$$

Stimo M

$$f(x) = \cos x \, e^{-x} \qquad f''(x) = 2 \sin x \, e^{-x} \qquad x \in [1, 10]$$

$$|f''(x)| = |2 \sin x \, e^{-x}| = 2\, |\sin x|\, |e^{-x}|$$

$$\leq 2 e^{-x} \leq 2 e^{-1} \leq 0{,}74$$

$$n \geq \sqrt{\frac{9^3 \; 0{,}74}{12 \; 10^{-6}}} = 6704{,}84 \qquad n \approx 6705$$

STIMA CON SIMPSON

$$E = -\frac{h^4}{180}(b-a)\, f^{IV}(\xi)$$

$$|E| = \frac{h^4}{180}(b-a)\, |f^{IV}(\xi)| \leq \frac{(b-a)^5}{180\, n^4} M \leq \varepsilon \qquad M \geq \underset{z\in[a,b]}{|f^{IV}(z)|}$$

$$n^4 \geq \frac{(b-a)^5}{180\,\varepsilon} M \qquad n \geq \sqrt[4]{\frac{(b-a)^5}{180\varepsilon} M} = (b-a)\sqrt[4]{\frac{(b-a)M}{180\varepsilon}}$$

$$f^{IV}(x) = -4\cos x\, e^{-x}$$

$$|f^{IV}(x)| = 4\,|\cos x|\,|e^{-x}| \leq 4e^{-x} \leq 4e^{-1} \leq 1{,}472$$

$$n \geq 9\sqrt[4]{\frac{9 \cdot 1.472}{180\; 10^{-6}}} = 148{,}24 \qquad n \simeq \underline{\underline{150}}$$

PARI

ESEMPI DI STIME FATTE CON LA SCURE

$$f(x) = (1 + x\cos x)\, e^{-x}$$

$$f''(x) = e^{-x}\,(2x\sin x - 2\sin x - 2\cos x + 1)$$

$$\max_{x\in[-1,2]} |f''(x)| = \ ?$$

$$|f''(x)| \le e^{-x}\,|2x\sin x - 2\sin x - 2\cos x + 1|$$

$$\le e^{-x}\,(|2x\sin x| + |2\sin x| + |2\cos x| + 1)$$

$$\le e^{-x}\,(2|x|\,|\sin x| + 2|\sin x| + 2|\cos x| + 1)$$

$$\le e^{-x}\,(2|x| + 2 + 2 + 1) = e^{-x}\,(2|x| + 5)$$

$$\le e\,(2\cdot 2 + 5) = 9e$$

ULTIMO ESEMPIO $\quad x \in [1,2]$

$$f(x) = \frac{x\cos x + 1}{1+x^2} \qquad f''(x) = \frac{-\cos x\, x^5 + 2x^4 \sin x - 7x\cos x + 6x^2 - 2\sin x - 2}{(1+x^2)^3}$$

$$|f''(x)| \le \frac{|\cos x|\,|x^5| + 2|x|^4 |\sin x| + 7|x|\,|\cos x| + 6|x|^2 + 2|\sin x| + 2}{|1+x^2|^3} \quad \min = 2$$

$$\le |x|^5 + 2|x|^4 + 7|x| + 6|x|^2 + 2 + 2$$

$$\le 2^5 + 2\cdot 2^4 + 7\cdot 2 + 6\, 2^2 + 4 = \cdot\, -$$

STIMA ERRORE METODO TRAPEZI

$$\int_{x_{k-1}}^{x_k} f(x)\,dx = \frac{f(x_{k-1})+f(x_k)}{2}\,h + E$$

$$g(t) = \int_{x_{k-1}}^{x_{k-1}+t} f(x)\,dx - \frac{f(x_{k-1})+f(x_{k-1}+t)}{2}\,t - E\left(\frac{t}{h}\right)^p$$

$g(h)=0 \quad g(0) = 0 - 0 - E\,0 = 0$

Rolle $\Rightarrow \quad g'(\alpha) = 0 \qquad \alpha \in (0,h)$

$$g'(t) = f(x_{k-1}+t) - \frac{f(x_{k-1})+f(x_{k-1}+t)}{2} - \frac{t}{2} f'(x_{k-1}+t)$$
$$+ \left(\frac{p}{h}\right) E \left(\frac{t}{h}\right)^{p-1}$$

$g'(0) = 0 \Rightarrow$ Rolle

$$g''(t) = \cancel{f'(x_{k-1}+t)} - \frac{1}{2}\cancel{f'(x_{k-1}+t)} - \frac{1}{2}\cancel{f'(x_{k-1}+t)}$$

$$- \frac{t}{2} f''(x_{k-1}+t) + \left(\frac{p(p-1)}{h^2}\right) E \left(\frac{t}{h}\right)^{p-2}$$

$g''(\beta) = 0$ $\qquad \beta \in (0,\alpha) \subseteq (0,h)$

so $p=3$

$$g''(\beta) = 0 \qquad -\frac{\cancel{\beta}}{2} f''(x_{k-1}+\beta) + \frac{6}{h^2} E \frac{\cancel{\beta}}{h} = 0$$

$$E = -\frac{h^2}{12} f''(x_{k-1}+\beta)$$

X SIMPSON

$$g(t) = \int_{x_{2k-1}-t}^{x_{2k-1}+t} f(x)\,dx - \frac{t}{3}\Big(f(x_{2k-1}-t) + 4 f(x_{2k-1}) + f(x_{2k-1}+t)\Big) - E\left(\frac{t}{h}\right)^p$$

$g(0) = g(h) = 0$ usando Rolle 2/3 volte si ottiene la stima