Esercizi svolti sulla trasformata di Fourier

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Anno Accademico 2007/2008

Sommario

primo schizzo...

Tabella delle Trasformate		
$af(t)+bg(t)$	$a\tilde{f}(\lambda)+b\tilde{g}(\lambda)$	1
$f(t-a)$	$e^{-i\lambda a}\tilde{f}(\lambda)$	2
$e^{iat}f(t)$	$\tilde{f}(\lambda-a)$	3
$f(\alpha t)$	$\frac{1}{\alpha}\tilde{f}\left(\frac{\lambda}{\alpha}\right)$	4
$f'(t)$	$(i\lambda)\tilde{f}(\lambda)$	5
$(-it)f(t)$	$\frac{\mathrm{d}\tilde{f}(\lambda)}{\mathrm{d}\lambda}$	6
$(f\star g)(t)$	$c_1\,\tilde{f}(\lambda)\,\tilde{g}(\lambda)$	7
$\chi_{[-a,a]}(t)$	$\frac{1}{c_1}\frac{2\sin(a\lambda)}{\lambda}$	8
1	$\frac{2\pi}{c_1}\delta(\lambda)$	9
$\mathrm{sign}(t)$	$-\frac{2i}{c_1\lambda}$	10
$\sum_{k=-\infty}^{\infty}\delta(t-k)$	$\frac{2\pi}{c_1}\sum_{m=-\infty}^{\infty}\delta(\lambda-2m\pi)$	11
$e^{-\alpha t^2}$	$\frac{1}{c_1}\sqrt{\frac{\pi}{\alpha}}e^{-\frac{\lambda^2}{4\alpha}}$	12
$e^{-\alpha\lvert t\rvert}$	$\frac{1}{c_1}\frac{2\alpha}{\alpha^2+\lambda^2}$	13
$\sin(\alpha t)$	$\pi i\big(\delta(\lambda+\alpha)-\delta(\lambda-\alpha)\big)$	14

Tabella delle Trasformate		
$\cos(\alpha t)$	$\pi\Big(\delta(\lambda+\alpha)+\delta(\lambda-\alpha)\Big)$	15
$\dfrac{1}{\lvert t\rvert}$	$i\pi-\log(-\lambda^2)$	16
$\dfrac{i}{\pi\, t}$	$\operatorname{sign}(\lambda)$	17
$\chi_{[-a,a]}(t)\sin(\alpha t)$	$i\left(\dfrac{\sin(a(\lambda+\alpha))}{\lambda+\alpha}-\dfrac{\sin(a(\lambda-\alpha))}{\lambda-\alpha}\right)$	18
$\chi_{[-a,a]}(t)\cos(\alpha t)$	$\dfrac{\sin(a(\lambda+\alpha))}{\lambda+\alpha}+\dfrac{\sin(a(\lambda-\alpha))}{\lambda-\alpha}$	19
$u(t)\sin(\alpha t)$	$i\pi\dfrac{\delta(\lambda+\alpha)-\delta(\lambda-\alpha)}{2}+\dfrac{\alpha}{\alpha^2-\lambda^2}$	20
$u(t)\cos(\alpha t)$	$\pi\dfrac{\delta(\lambda+\alpha)+\delta(\lambda-\alpha)}{2}+\dfrac{i\lambda}{\alpha^2-\lambda^2}$	21
$u(t)e^{-at}\sin(\alpha t)$	$\dfrac{\alpha}{a^2+2ia\lambda+\alpha^2-\lambda^2}$	22
$u(t)e^{-at}\cos(\alpha t)$	$\dfrac{a+i\lambda}{a^2+2ia\lambda+\alpha^2-\lambda^2}$	23
$u(t)e^{-at}$	$\dfrac{1}{a+i\lambda}$	24
$u(t)t\, e^{-at}$	$\dfrac{1}{(a+i\lambda)^2}$	25
$u(t)t^k\, e^{-at}$	$\dfrac{k!}{(a+i\lambda)^{k+1}}$	26

1 Esercizi svolti

1.1 Equazione del calore 1D

Equazione del calore unidimensionale su dominio infinito

$$\begin{aligned}\frac{\partial T(t,x)}{\partial t} &= K\frac{\partial^2 T(t,x)}{\partial x^2}, \qquad t>0, \quad x\in\mathbb{R}\\ T(0,x) &= T_0(x), \qquad x\in\mathbb{R}\end{aligned}$$

applicazione della trasformata di Fourier per la coordinata x

$$\frac{\partial \tilde{T}(t,\lambda)}{\partial t} = -K\lambda^2 \tilde{T}(t,\lambda)$$

questa equaizione differeziale e molto semplice ed ha soluzione

$$\tilde{T}(t,\lambda) = \tilde{T}_0(\lambda) e^{-K\lambda^2 t}$$

possiamo ora antitrasformare usando il prodotto di convoluzione (regola 7)

$$T(t,x) = \frac{1}{c_1} \int_{-\infty}^{\infty} T_0(t-z) g(t,z) \, \mathrm{d}z \tag{1}$$

dove

$$g(t,x) = \mathcal{F}^{-1}\left\{ e^{-K\lambda^2 t} \right\}.$$

Usiamo ora la regola 12 con

$$\frac{\lambda^2}{4\alpha} = K\lambda^2 t$$

da cui otteniamo $\alpha = 1/(4Kt)$ e moltiplicando per $c_1 \sqrt{\alpha/\pi}$ otteniamo

$$g(t,x) = \frac{c_1}{\sqrt{4\pi K t}} e^{-x^2/(4Kt)}$$

e sostituendo in (1) otteniamo

$$\begin{aligned} T(t,x) &= \frac{1}{2\sqrt{\pi K t}} \int_{-\infty}^{\infty} T_0(x-z) e^{-z^2/(4Kt)} \, \mathrm{d}z \\ &= \frac{1}{2\sqrt{\pi K t}} \int_{-\infty}^{\infty} T_0(w) e^{-(x-w)^2/(4Kt)} \, \mathrm{d}w \end{aligned}$$

Esempi di soluzioni particolari:

1. Se $T_0(x) = \delta(x)$ allora otteniamo

$$\begin{aligned} T(t,x) &= \frac{1}{2\sqrt{\pi K t}} \int_{-\infty}^{\infty} \delta(x-z) e^{-z^2/(4Kt)} \, \mathrm{d}z \\ &= \frac{e^{-x^2/(4Kt)}}{2\sqrt{\pi K t}} \end{aligned}$$

2. Se $T_0(x) = \mathcal{X}_{[-1,1]}(x)$ allora otteniamo

$$
\begin{aligned}
T(t,x) &= \frac{1}{2\sqrt{\pi K t}} \int_{-\infty}^{\infty} \mathcal{X}_{[-1,1]}(w) e^{-(x-w)^2/(4Kt)}\, \mathrm{d}w \\
&= \frac{1}{2\sqrt{\pi K t}} \int_{-1}^{1} e^{-(x-w)^2/(4Kt)}\, \mathrm{d}w
\end{aligned}
$$

3. Se $T_0(x) = T_0 \mathrm{sign}(x)$ allora otteniamo

$$
\begin{aligned}
T(t,x) &= \frac{1}{2\sqrt{\pi K t}} \int_{-\infty}^{\infty} T_0 \mathrm{sign}(w) e^{-(x-w)^2/(4Kt)}\, \mathrm{d}w \\
&= \frac{T_0}{2\sqrt{\pi K t}} \int_{0}^{\infty} e^{-(x-w)^2/(4Kt)}\, \mathrm{d}w - \frac{T_0}{2\sqrt{\pi K t}} \int_{-\infty}^{0} e^{-(x-w)^2/(4Kt)}\, \mathrm{d}w \\
&= \frac{T_0}{2\sqrt{\pi K t}} \int_{0}^{\infty} e^{-(x-w)^2/(4Kt)}\, \mathrm{d}w - \frac{T_0}{2\sqrt{\pi K t}} \int_{0}^{\infty} e^{-(x+w)^2/(4Kt)}\, \mathrm{d}w \\
&= \frac{T_0}{2\sqrt{\pi K t}} \int_{0}^{\infty} \left[e^{-(x-w)^2/(4Kt)} - e^{-(x+w)^2/(4Kt)} \right] \mathrm{d}w
\end{aligned}
$$

per questa soluzione calcoliamo la sua derivata

$$
\begin{aligned}
\frac{\mathrm{d}T(t,x)}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t} \frac{T_0}{2\sqrt{\pi K t}} \int_{0}^{\infty} \left[e^{-(x-w)^2/(4Kt)} - e^{-(x+w)^2/(4Kt)} \right] \mathrm{d}w \\
&= \frac{T_0}{2\sqrt{\pi K t}} \int_{0}^{\infty} \frac{\mathrm{d}}{\mathrm{d}t} \left[e^{-(x-w)^2/(4Kt)} - e^{-(x+w)^2/(4Kt)} \right] \mathrm{d}w \\
&= \frac{T_0}{2\sqrt{\pi K t}} \int_{0}^{\infty} \frac{2}{4Kt} \left[(x-w) e^{-(x-w)^2/(4Kt)} - (x+w) e^{-(x+w)^2/(4Kt)} \right] \mathrm{d}w \\
&= \frac{T_0}{4(Kt)^{3/2}\sqrt{\pi}} \int_{0}^{\infty} \left[(x-w) e^{-(x-w)^2/(4Kt)} - (x+w) e^{-(x+w)^2/(4Kt)} \right] \mathrm{d}w
\end{aligned}
$$

per $x = 0$ abbiamo

$$
\begin{aligned}
\left.\frac{\mathrm{d}T(t,x)}{\mathrm{d}t}\right|_{x=0} &= -\frac{T_0}{2(Kt)^{3/2}\sqrt{\pi}} \int_{0}^{\infty} w e^{-w^2/(4Kt)}\, \mathrm{d}w \\
&= -\frac{T_0}{2(Kt)^{3/2}\sqrt{\pi}} \left[-2Kt e^{-w^2/(4Kt)} \right]_0^{\infty} \\
&= -\frac{T_0}{\sqrt{\pi K t}}
\end{aligned}
$$

1.2 Età della terra

T_0	2000	$[C^\circ]$	Temperatura iniziale
dT/dx	0.037	$[C^\circ/m]$	Variazione della temperatura
K	$1.2\,10^{-6}$	$[m^2/s]$	Conduttività termica
R	6400	$[Km]$	Raggio della terra